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THE BARGAINING SET AND NONSTANDARD ANALYSIS.(U)

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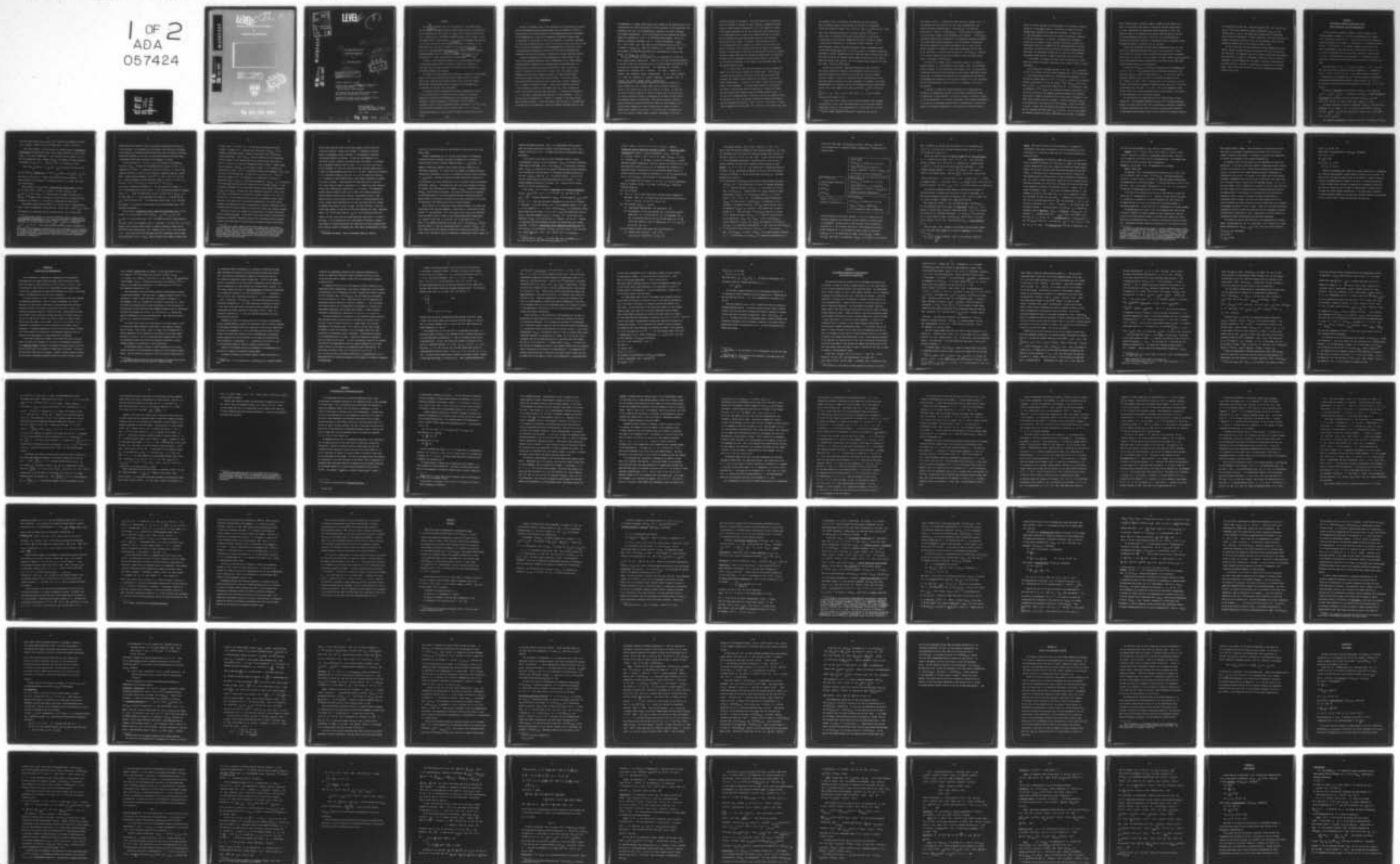

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THE BARGAINING SET AND NONSTANDARD ANALYSIS.

11 Jun 78

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by

10 John Geanakoplos

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Technical Report No. 1

Prepared under Contract No. 15 N00014-77-C-0533
Project No. NR 277-240
for the Office of Naval Research

Harvard University
1737 Cambridge Street, Room 404
Cambridge, Massachusetts 02138

June, 1978

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ABSTRACT

We give an elementary proof of the equivalence of the Bargaining Set and the Core is given for nonstandard infinite economic games based on a new definition of the Bargaining Set that retains the spirit of the original definition and which is appropriate to the "large economy" nonatomic model of Aumann and the nonstandard model of Brown-Robinson. As a corollary ~~we derive~~ a generalization of the result obtained by Lloyd Shapley is derived concerning the convergence of the Bargaining Set to the Core in sequences of replicated economic games. In particular, ~~we do not need to restrict our~~ attention is not restricted to replicated sequences, ~~we do not assume~~ concave utilities are not assumed, and ~~we drop~~ the restriction of the Bargaining Set to pareto optimal payoff configurations is dropped.

We give a self contained introduction to the new mathematical theory of nonstandard analysis which is designed to enable the economist to critically read this paper and many of the others in the literature of nonstandard mathematical economics.

We compare the three mathematical (replicated, nonatomic, and nonstandard) models of the central hypothesis of perfect competition that one trader alone cannot influence the conditions faced by him or any other trader, arguing that the nonstandard model is conceptually and mathematically the simplest and the most free from paradox.

We attempt to give a straightforward intuitive explanation of why it is to be expected that all the great solution concepts, core, values, competitive equilibrium, and Bargaining Set, are equivalent when the hypothesis of perfect competition is maintained. We also present a systematic account of the relationship between an economy $E = (t, I(t), \succ_t)_{t \in T}$ and the derived transferable utility economic games $(T, V, E. [U_t]_{t \in T})$, indicating perhaps why it may be useful to study the latter as well as the former.

INTRODUCTION

We give an elementary proof of the equivalence of the Bargaining Set and the Core for Nonstandard infinite economic games, and as a Corollary we derive a generalization of the result obtained by Lloyd Shapley concerning the convergence of the Bargaining Set to the Core in sequences of replicated economic games. In particular, we do not need to restrict our attention to replicated sequences, we do not need to assume that the utility functions are concave, and lastly we show that we can drop the assumption that the Bargaining Set be restricted to pareto optimal payoff configurations, and still we find that the Bargaining Set converges to the Core as the number of traders in an economy approaches infinity.

We give a new definition of the Bargaining Set, closely related to the spirit of the original definition, that is appropriate for the "large economy" *nonatomic model of* Aumann and the nonstandard model of Brown-Robinson. The Bargaining Set was invented by Aumann and Maschler in 1963 in an effort to capture the payoffs that could occur in a game (T,V) when agents actually bargained with each other, weighing the desire for higher payoffs against the risk that excessive greed would bring reprisals from the other traders. The precise definition was based on the intuition that a payoff configuration $[x_t]$ is stable iff whenever a trader i sees a way of forming a coalition S that promises to pay more to each member than $[x_t]$ did, he neglects to "start the revolution" for fear that some trader j, alarmed by the formation of S, will form a counterobjecting coalition R that guarantees j his x_j and also any of the members of S used in R whatever i had offered them, thus potentially leaving i alone and unable to get even the x_i he began with. We explain why the old definition, despite the Shapley convergence theorem, with

its dependence on a single trader acting as the "leader" is not appropriate for large economies, even for replicated sequences of finite economies. In one sense, rather than shrinking to the core, the old Bargaining Set expands in the limit to include every payoff configuration. In the new definition, we have replaced the solitary leader of an objection by a set K of arbitrarily small measure. That is, an objection, instead of specifying a set S, a payoff configuration $[y_t]_{t \epsilon S}$ for S and a leader $t_0 \epsilon S$ must under our new definition specify for any $\delta > 0$ a triplets $(S, [y_t]_{t \epsilon S}, K)_\delta$ where the "measure" of K is less than δ, $K \subset S$, and in order to be justified there must be no counterobjection to any of the triplit $(S, [y_t]_{t \epsilon S}, K)_\delta$ for all $\delta > 0$.

We have also dropped the requirement that the Bargaining Set contain only pareto optimal payoffs (imputations). Indeed one of the most interesting questions we can ask is under what conditions can we be sure that bargains and contracts will be struck so as to produce imputations. The Bargaining Set provides a framework for analyzing such a question; for small economies it contains many suboptimal payoff configurations, but in large economies (i.e. when the conditions of perfect competition rigorously hold) it contains only pareto optimal payoff configurations.

nonstandard model of Brown-Robinson rigourize the idealized assumption of perfect competition, that a single trader cannot by himself influence the terms of trade faced by him or any other trader in the market. We show that the notion of perfect competition, like the original idea of differentiation in calculus conceived by Leibnitz, presupposes the existence of agents which are negligibly small but nonvanishing. This apparent paradox is solved by modern mathematical economics with measure theory and limit arguments; thus in 1964 Aumann suggested as a model a measure space of agents in which a single trader, thought of as a point t on the real line, has measure zero and so his presence or absence makes literally no difference to the total

resources available to the market. Yet in what sense such a trader even exists is difficult to explain; he seems a phantom. Nonstandard analysis solves the same paradox by extending the set of real numbers to include infinitesimals satisfying all the same properties as the ordinary reals. In the nonstandard model a trader is an infinitesimal part of the market and can therefore only have infinitesimal effects on the conditions affecting the other traders. We include a self-contained account of all the mathematical theory necessary to critically read this paper and many of the others in the economic literature that depend on nonstandard analysis. We take pains to show that the theorems which hold in the measure theoretic economic model and the nonstandard economic model are completely analogous, but the mathematical sophistication required to prove them, far from being overwhelming, is much <u>less</u> for the nonstandard model than the measure theoretic model.

It is a curious fact that when the hypothesis of perfect competition holds, all of the solution concepts for economic games $(T,V,E,[U_t]_{t\epsilon T})$, no matter what principles of allocation they are based on provided they recognize the original private ownership of all property, give the same payoff configurations. In Section IV we present a simple intuitive explanation of this phenomenon. In economic games without a central planner and without a coordinating system of market prices, traders must bargain with and threaten other traders to gain the maximum possible payoff x_t . The only direct bargaining power a trader has, and the only immediate threat he can make, is to quit the economy, removing his resources from general use, thus causing a total loss of $V(T) - V(T\sim[t])$. Each trader t would like to demand at least the payoff $V(T) - V(T\sim[t])$; if he doesn't get it he

can threaten to leave the economy. The difficulty is that for finite games in general (always if the utilities are concave) it is impossible to give each trader the payoff $V(T) - V(T\text{~}[t])$, that is $\sum_{t\varepsilon T} [V(T)-V(T\text{~}[t]] > V(T)$. Hence some simplifying principle of allocation, whether it is based on power (core), or fairness (value payoff configurations), or some notion of what makes a viable contract (Bargaining Set) must be invoked to achieve a stable payoff configuration in an apparently chaotic situation. All of these principles, and the free market allocation as well, essentially reduce to the payoff configuration $[V(T) - V(T\text{~}[t])]_{t\varepsilon T}$ when it is feasible and when the hypothesis of perfect competition hold (of course this must be shown in each case). The riddle of why all the solution concepts approach each other in the limit is mainly the riddle of why $\sum_{t\varepsilon T} [V(T) - V(T\text{~}[t])]$ becomes nearly feasible as $|T|$ gets very large. And this we can explain.

We show that the perfect competition hypothesis is made concrete in every model (replicated, continuum, and nonstandard) by what we call the essential assumption that traders are more alike than they are different. The set of traders is taken to be infinite while the characteristics of the individual agents (utilities and initial endowments) is assumed to be bounded. Mathematically, the consequence is that the convexifying effect of many traders and the law of large numbers can be used to show that $\sum_{t\varepsilon T} [V(T) - V(T\text{~}[t])] \simeq \sum_{K=1}^{|T|} [V(t_1,\ldots,t_K) - V(t_1,\ldots t_{K-1})] = V(T)$ where $t_1,\ldots,t_{|T|}$ is a random ordering of the traders in $|T|$.

We show that for the version of the essential assumption used by Aumann and Brown-Loeb in their value equivalence papers and in this paper also,[1] any nonstandard economy E can be approximated by a type economy $\bar{E}$ (in which

[1] They all assume "bounded differentiable" preferences and utilities.

each trader is one of v types and the ratio of types to traders, $v/|T|$, is infinitesimal) in such a way that the core, value, Bargaining Set, and competitive payoff configurations of $(T,V,E,[U_t]_{t\epsilon T})$ are identical, respectively, to those in $(T,\bar{V},\bar{E},[\bar{U}_t]_{t\epsilon T})$. Hence proving the equivalence of those solution concepts in all nonstandard type economies $(T,\bar{V},\bar{E},[\bar{U}_t]_{t\epsilon T})$ immediately implies their equivalence in all nonstandard economies satisfying the bounded differentiable assumptions. Thus the sequence of (unequally) replicated economies of Scarf-Debreu and Shapley are in a sense the most general sequences of economies for which these equivalences can be proved. On the other hand, it turns out that the proofs in $(T,V,E,[U_t]_{t\epsilon T})$ directly are at least as simple as those which attempt to exploit the type economy nature of the games $(T,\bar{V},\bar{E},[\bar{U}_t]_{t\epsilon T})$.

In Section 2 we attempt to give a systematic account of the relation between economies E, nontransferable utility games $(T,\vec{V})$ *and transferable* utility economies $\bar{E}$ and transferable utility games (T,V), and allocations, transferable utility allocations and what we call natural transferable utility allocations. We introduce the definition of the Bargaining Set and show how the Bargaining Set equivalence is the natural successor to the core equivalence and the value equivalence (later we show it is a consequence of the value equivalence).

In Section 3 we amend the original definition of the Bargaining Set, showing how to preserve the intuition of the original definition and making it appropriate to large economies. In Section 4 we give our heuristic account of all the equivalences. In Section 5 we present a self contained introduction to the subject of nonstandard analysis which is designed both to convince the reader of its essential simplicity and to allow him to critically read the

rest of this paper and many of the other papers in the economic literature making use of nonstandard analysis. In Section 6 we show how the nonstandard model rigourizes the competitive hypothesis. We also prove our type approximation theorem which we suggest is practically equivalent to the value equivalence. In Section 7 we summarize the differences between the three models of large economies (replicated, continuum, and nonstandard) and we try to suggest that the nonstandard model is the most appropriate. Sections 8 and 9 are devoted to proving the Bargaining Set equivalence for nonstandard economies and for limits of finite economies, respectively, while Section 10 suggests possibilities for further study.

It is possible to summarize the major ideas of this paper in a small space and to point out several connections to the classical theory of perfect competition.

Let us begin with an exchange economy $E = (T, I(t), >_t)_{t \in T}$ consisting of a set T of traders t who have initial endowments $I(t) \in \mathbb{R}^\ell$ and preferences $>_t$. By choosing utility functions U_t representing the preferences $>_t$ we can form an economic game $(T, V) = (T, V. E, [U_t]_{t \in T})$ where V is a function defined on the subsets S of T by the formula: $V(S) = \mathrm{Max} \sum_{t \in S} U_t(X(t))$ such that $\sum_{t \in S} X(t) \leq \sum_{t \in S} I(t)$ and $X(t) \in \mathbb{R}^\ell_t$ for all $t \in S$. It is fruitful to think of the game $(T, V., E, [U_t]_{t \in T})$ as a production economy in which there is a fixed set of producers t who organise production through various bargains and contracts, shipping raw materials I(t) to each other in exchange for the final output $U_t(X(t))$ (of food or energy). A payoff configuration $[X_t]_{t \in T}$ is an allocation of the final output to each producer. There is no reason to expect, a priori, that the bargains and contracts will be made in such a way as to maximize the total output $\sum_{t \in T} X_t$ of food. Yet if we also embody the essential postulate of perfect competition into our model, by assuming

that T contains nearly an infinite number of agents and that almost every agent t can find many other agents t' with nearly the same characteristics (so that a producer with an essentially unique technology is rare), then indeed not only do the competitive allocations, value allocations, and core allocations maximize total output of food, but so do the Bargaining Set allocations; furthermore these sets of allocations, based on apparently widely diverging principles of allocation, are all the same.

Observe that underlying any final payoff configuration $[X_t]_{t\epsilon T}$ of food there is an underlying allocation $[X(t)]_{t\epsilon T}$ of the raw materials. Only by chance will it be the case that each producer keeps exactly what he produces, that is that $X_t = U_t(x(t))$. For the economy as a whole, however, $\sum_{t\epsilon T} X_t = \sum_{t\epsilon T} U_t(X(t))$. Clearly many different payoff configurations could arise from the same underlying allocation $[X(t)]_{t\epsilon T}$ of raw materials.

We show that if $[X_t]_{t\epsilon T}$ is any payoff configuration in the Bargaining Set of (T, V) = (T, V, E, $[U_t]_{t\epsilon T}$) arising from an underlying allocation of raw materials $[X(t)]_{t\epsilon T}$, then we can find prices $(p_1, \ldots, p_\ell)$ for each of the raw materials (taking the price of food as 1) such that for almost all $t\epsilon T$, $X_t \simeq U_t(X(t)) - p^t\,(X(t) - I(t))$. Moreover, even if the functions U_t are not concave, if the perfectly competitive hypothesis holds, then $U_t(y) - p^t(y-I(t))$ is maximized, over all $y\epsilon \mathbb{R}^\ell_t$. at $y = x(t)$. This is the fundamental mathematical result, due to the convexifying effect of large numbers, that accounts for the equivalence of all the solution concepts.

We note several of the implications for the classical theory of pure competition. The fundamental mathematical result of the preceding paragraph implies, as Sraffa noted in 1926, that in equilibrium, assuming the conditions of perfect competition hold, every firm must be operating under conditions of decreasing (nonincreasing) returns to scale, even if his production function

is not generally of that form. From the equalities $V(T) - V(T \sim[t]) \simeq X_t$ and $\sum_{t\epsilon T} X_t \simeq V(T)$ for any competitive payoff configuration $[X_t]_{t\epsilon T}$, we can conclude that J. B. Clark was right when he claimed that in equilibrium, again assuming the hypothesis of perfect competition, that every agent paid his marginal contribution, despite the fact (as Joan Robinson noted), that the production functions may not show constant returns to scale.

We note at the last that the hypothesis of perfect competition is often taken to include a second postulate concerning the free entry of new firms into the market. By assuming a fixed set of agents we have explicitly ignored this possibility; it might be interesting to extend our model to allow for investment. We could imagine that each agent t could use a fixed amount of raw materials $\overline{Z}(t)$ to construct a second plant U_t identical to his original plant. Then if $\overline{Z}(t)$ were very small, each firm in equilibrium would make profits close to zero.

SECTION II

Game Theory, Competitive Equilibria, Core, Value Allocations, and the Bargaining Set

In this section we attempt to present a unified treatment of the relation between an economy E and all the transferable utility games (T,V) which can be derived from E. We explore in detail the connection between allocations in an economy E, allocations in a transferable utility economy $\bar{E}$ obtained from E by adding an imaginary extra good called money ("transferable utility allocations"), allocations for both E and $\bar{E}$ (natural transferable utility allocations), and payoff configurations in (T,V). We indicate why we are interested in transferable utility games and we introduce all the solution concepts we shall be concerned with later.

Our basic model is a set T of traders or agents $T = [1,2,\ldots W]$, such that with every trader we associate initial endowments of ℓ commodities, $[I(1),\ldots,I(W)]$, $I(t) \varepsilon R^{\ell}$ for all t and ordinal preference relations $>_t$ defined over the consumption set R^{ℓ}_{+} which we assume are representable by continuously differentiable monotonic utility functions $U_t\colon R^{\ell}_{+} \to R$. Our economy is therefore essentially described by the notation $E = [(t, I(t), >_t)_{t\varepsilon T}]$.

We define an allocation as any division $[x(t)]_{t\varepsilon T}$ of the resources $\sum_{t\varepsilon T} I(t)$, $\sum_{t\varepsilon T} x(t) = \sum_{t\varepsilon T} I(t)$. An allocation $[x(t)]_{t\varepsilon T}$ is Pareto Optimal for the Economy E iff there is no other allocation $[y(t)]_{t\varepsilon T}$ such that $y(t) \gtrsim_t x(t)$ for all $t\varepsilon T$ and for some $t_0\varepsilon T$, $y(t_0) >_{t_0} x(t_0)$.

The core is defined as the set of allocations $[x(t)]_{t\varepsilon T}$ such that for no coalition S, i.e. subset S of T, is there a suballocation $[y(t)]_{t\varepsilon S}$ such that $\sum_{t\varepsilon S} y(t) = \sum_{t\varepsilon S} I(t)$ and $y(t) \gtrsim_t x(t)$ for all $t\varepsilon S$ and for some $t_0 \varepsilon S$, $y(t_0) >_{t_0} x(t_0)$.

The competitive equilibria are defined as the set of allocations $[x(t)]_{t\varepsilon T}$

such that there exists a price vector $p \epsilon R^{\ell}$ which has the property that there is no $y \epsilon B_t(p) = [y \epsilon R^{\ell} | y \geqq 0,\ p^t y \leqq p^t I(t)]$ with $y >_t x(t)$ for any $t \epsilon T$.

With every economy E we can associate an infinite number of <u>transferable utility games</u> in characteristic form. Such a game is defined as a pair (T,V) where T is a set of traders and V is a function from the subsets S of T into the nonnegative reals. Suppose $[U_t]_{t\epsilon T}$ is a family of utility functions representing $[>_t]_{t\epsilon T}$ in E. Then let $(T,V) = (T,V,E,[U_t]_{t\epsilon T})$ where

$$V(S) = \text{Max} \sum_{t\epsilon S} U_t(x(t)) \text{ s.t. } \sum_{t\epsilon S} x(t) = \sum_{t\epsilon S} I(t) \text{ and } x(t) \geqq 0 \text{ for all } t\epsilon S .$$

We call $[\bar{x}(t)]_{t\epsilon S}$ <u>maximal for S</u> if $V(S) = \sum_{t\epsilon S} U_t(\bar{x}(t))$ and $\sum_{t\epsilon S} \bar{x}(t) = \sum_{t\epsilon S} I(t)$ and $\bar{x}(t) \geqq 0$ for all $t\epsilon S$. Observe that with a finite set of agents, V is always defined as the maximum of a continuous function on a compact set, hence it is well-defined.

More generally, one can define a <u>nontransferable utility game</u> by the pair $(T, \vec{V})$ where $\vec{V}(S) = [z \epsilon R^{|S|} | z_t \leqq U_t(x(t))$ for some $[x(t)]_{t\epsilon S}$ with $\sum_{t\epsilon S} x(t) \leqq \sum_{t\epsilon S} I(t)]$ where $|S|$ simply denotes the number of traders in coalition S. Evidently $\vec{V}(S)$ indicates the various levels of satisfaction which the members of coalition S can achieve by cooperating with each other.*

Transferable utility games evidently are a special class of nontransferable games: given a transferable utility game (T,V) define $\vec{V}(S) = [z \epsilon R_+^{|S|} | \sum_{t\epsilon S} z_t \leqq V(S)]$. The assumption of transferable utility may seem drastic, but it allows great

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*It is important to point out, as Scarf already has (see for instance Scarf: <u>The Computation of Equilibria</u> P. 203), that our definitions of transferable and nontransferable economic games could be extended to allow for a common production technology Y, so that we would require $\sum_{t\epsilon S} x(t) - \sum_{t\epsilon S} I(t) \ \epsilon\ Y$ instead of $\sum_{t\epsilon S} x(t) - \sum_{t\epsilon S} I(t) \leqq 0$. "The conventional neoclassical convexity assumptions on preferences and technological possibilities are quite unnecessary for the definition of the game. This observation raises the hope that some of the important variations of the neoclassical model, such as increasing returns to scale in production, may be capable of analysis in game-theoretic terms rather than by more conventional behavioristic assumptions."

simplification and without it some of the solution concepts we use could not even be defined. In this paper we are following a long tradition by confining our attention to transferable utility games; while we do not propose to formally defend this tradition, we can say a few words describing more precisely what a transferable utility game is and why it may sometimes be useful.

Given the economy $E = (t, I(t), >_t)_{t\varepsilon T}$ and a family of representing utilities $[U_t]_{t\varepsilon T}$ consider the transferable utility economy $\bar{E}$ with $\ell + 1$ commodities, the first ℓ of which are identical to those in E, and the last commodity, which may be held in either positive or negative quantities, called money. Define initial endowments $\bar{I}(t) \varepsilon R^{\ell+1}$ by $\bar{I}(t) = (I(t), 0)$, for all $t\varepsilon T$. Define preferences from the utilities $\bar{U}_t(x,\xi) = U_t(x) + \xi$, $x \varepsilon R^{\ell}$, $\xi \varepsilon R$. The game $(T,V) = (T,V,E,[U_t]_{t\varepsilon T})$ derived from E and $[U_t]_{t\varepsilon T}$ can be seen as completely describing $\bar{E}$. For any coalition of traders S, $V(S)$ is the maximum sum of total utility S can collectively guarantee its members. Since in $\bar{E}$ money is transferable, any set of utility levels $[x_t]_{t\varepsilon S}$, one for each trader in S, can be guaranteed the traders in S provided the sum $\sum_{t\varepsilon S} x_t$ does not exceed $V(S)$. The transferable utility game (T,V) describes the economy $\bar{E}$ in exactly the same fashion as the nontransferable game $(T,\vec{V})$ describes the economy E.

We can define a <u>transferable utility competitive equilibrium (tuce)</u> for the economy $\bar{E}$ as an allocation $[x(t), \xi(t)]_{t\varepsilon T}$, $\sum_{t\varepsilon T} x(t) = \sum_{t\varepsilon T} I(t)$ and $\sum_{t\varepsilon T} \xi(t) = 0$, and a price vector $p \varepsilon R^{\ell}$ (we assume the price of money equals 1) such that $(x(t), \xi(t))$ maximizes $\bar{U}_t(x, \xi)$ such that $x \geqq 0$ and $p^t x + \xi \leqq p^t I(t)$. However, since we allow either positive or negative holdings of money this is equivalent to the condition that $x(t)$ maximize $U_t(x) - p^t(x - I(t))$ such that $x \geqq 0$. Now it may turn out by coincidence that the transferable utility competitive allocation $[x(t), \xi(t)]_{t\varepsilon T}$ does not require the transfer of money, that

is that $\xi(t) = 0$ for all $t \epsilon T$. In that case the transferable utility economy $\bar{E}$ singles out in a natural way an allocation $[x(t)]_{t\epsilon T}$ in the original conventional economy. Similarly there may be other transferable utility allocations $[y(t), \eta(t)]_{t\epsilon T}$ which are important in the transferable utility economy $\bar{E}$ (for instance the value, which we define later) and if $\eta(t) = 0$ for all $t\epsilon T$, then the transferable utility economy $\bar{E}$ again singles out an allocation $[y(t)]_{t\epsilon T}$ in the original economy E in a natural way. Observe that a trader t in $\bar{E}$ is indifferent between the assignments $(x(t), \xi(t))$ and $(y(t), \eta(t))$ provided that $U_t(x(t)) + \xi(t) = U_t(y(t)) + \eta(t)$. Hence so far as the traders in $\bar{E}$ are concerned, it is sufficient to describe an allocation $[x(t), \xi(t)]_{t\epsilon T}$ by the payoff configuration $[x_t]_{t\epsilon T}$, $x_t = U_t(x(t)) + \xi(t)$. If we were concerned only with the allocations in the original economy E, then we would be interested in the relationship between various payoff configurations $[x_t]_{t\epsilon T}$ and $[y_t]_{t\epsilon T}$ provided that they represented allocations $(x(t), \xi(t))$ and $(y(t), \eta(t))$ which did not involve the transfer of money, $\xi(t) = \eta(t) = 0$ for all $t\epsilon T$. In this paper we shall prove that the competitive payoff configurations are the same as the value payoff configurations, core payoff configurations, and Bargaining Set payoff configuations in infinite transferable utility economies (even if they do as well as if they do not involve the transfer of money).

There are at least four reasons that we might want to consider transferable utility economies and games. In the first place, aside from the relatively minor possibility of negative consumption of money, the transferable utility exchange economy we just described is a special kind of exchange economy;* results which

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*If there is a single commodity which enters linearly into each trader's utility function, that is every trader has a constant marginal utility for some good (say money), then the original economy E is itself a transferable utility economy. Classical authors, such as Marshall made this assumption and recently Truman Bewley has argued that traders maximizing utility over a many period horizon with uncertain future prices may act in any period as if they had a constant marginal utility of money.

hold for the latter must hold for the former simpler case for which the proofs may be easier, Second, every economy E determines an infinite number of transferable utility economies $\bar{E}$ corresponding to different representing families of utilities. We shall see that sometimes it is sufficient to prove a difficult proposition about the economy E by showing that an analogous property holds for every one of the derived transferable utility economies $\bar{E}$, where the proof may be easy, and then arguing that as a result the same proposition must hold for E as well. Third, we can interpret $\bar{E}$ as a production economy in which the U_t's are the production functions of W producers of some finished product, which might conveniently be thought of as food or energy.* Some producers own large quantities of resources $I(t)$ while others are adept at producing a lot of energy with fewer resources because their *technological possibilities* U_t are better. The fundamental question which we examine is what sort of deals will be made between well-endowed producers and efficient producers; in particular we want to know whether without a central planner and even without a coordinating market and prices, selfish agents interested only in their own welfare will act together to maximize world output of food or energy. We shall find that if there are a great many traders, so that none of them owns originally more than a negligible amount $I(t)$ of the total resources $\sum_{t \in T} I(t)$ and if production plants are limited, so that $U_t(x)$ does not exceed some fixed upper bound M no matter how large x is (a producer may own a fixed number of factories or employ a fixed number of trained technicians) then indeed the only secure bargains which will be struck, called the Bargaining Set, will assure the maximization of world

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*See Aumann and Shaply: Values of Nonatomic Games, pp. 180-181.

output and in fact will assure the same division of final output that a free market would.

Another interpretation of $\bar{E}$, and our fourth reason for considering transferable utility economies and games, corresponds to the problem of a central planner trying to maximize community welfare $W = \sum_{t\varepsilon T} U_t(x(t))$ (we imagine an additively separable social welfare function consistent with individual preferences). Of course the social planner is necessarily making interpersonal comparisons of utility; by considering different welfare functions $W = \sum_{t\varepsilon T} \lambda_t U_t(x(t))$ obtained by varying the weight (importance) attributed to each individual, the social planner will be led to consider different maximizing allocations $[x(t)]_{t\varepsilon T}$. The planner may wish to set more stringent criterion, for instance that consistent with the weights $[\lambda_t]_{t\varepsilon T}$, a maximizing allocation $[x(t)]_{t\varepsilon T}$ also be fair in some sense. This idea of fairness can be rigourized by the precise notion of the value allocation defined by Shapley and Aumann. Again we shall be interested in finding out that these allocations are the same as those which a free market would provide in large economies.

We now return to our definitions, giving precise meaning to the terms used intuitively in the last few paragraphs. Recall that in a transferable utility economy $\bar{E}$, two allocations $(x(t), \xi(t))_{t\varepsilon T}$ and $(y(t), \eta(t))_{t\varepsilon T}$ are equivalent (from the traders' point of view) if the corresponding payoff configurations $x_t = U_t(x(t)) + \xi(t)$ and $y_t = U_t(y(t)) + \eta(t)$ as the same for all $t\varepsilon T$. Hence $\bar{E}$ is completely described by the game $(T,V) = (T,V,E,[U_t]_{t\varepsilon T})$ derived from the economy E with the representing family of utilities $[U_t]_{t\varepsilon T}$. As a final remark we note that Shapley and Shubik have answered the question, given a game (T,V) when can we be sure it describes a transferable utility economy $\bar{E}$?

Shapley and Shubik proved that (T,V) is a transferable utility economic game with concave utilities if an only if the restricted game (S,V) has a nonempty core for all $S \subset T$. We define the core of a transferable utility game below.

It is crucial to note that if we use a different family of utility representations $[U_t]_{t\epsilon T}$ for the preferences $[>_t]_{t\epsilon T}$ we will get a different transferable utility game $(T,V) = (T,V,E,[U_t]_{t\epsilon T})$. In particular, given one family $[U_t]_{t\epsilon T}$ of utilities we can construct infitely many different families, hence infitely many different games, by considering representing families of the form $[\lambda_t U_t]_{t\epsilon T}$ where the λ_t's are positive scalars. In fact one of the most fruitful means of discovering properties of the economy E is to examine analogous properties of the games (T,V) derived from the various families of utility representations.

Given a game (T,V) we define an <u>imputation</u> as any <u>payoff configuration</u> $[x_t]_{t\epsilon T}$, $\sum_{t\epsilon T} x_t \leq V(T)$, $x_t \epsilon R$, $x_t \geq 0$ for all $t\epsilon T$ such that in fact $\sum_{t\epsilon T} x_t = V(T)$. A payoff configuration $[x_t]_{t\epsilon T}$ is in the <u>core</u> iff $\sum_{t\epsilon S} x_t \geq V(S)$ for all $S \subset T$. Note that any payoff configuration in the core is a fortiori an imputation. We use the notation $x(t)$ to be a commodity bundle in R^ℓ, and x_t to be a payoff in R. If $(T,V) = (T,V,E,[U_t]_{t\epsilon T})$ is a transferable utility game arising from the economy $E = ((t, I(t), >_t)_{t\epsilon T})$ and representing utilities $[U_t]_{t\epsilon T}$ then every maximal allocation $[x(t)]_{t\epsilon T}$ for T gives rise to an imputation $[x_t]_{t\epsilon T}$ in the natural way $x_t \equiv U_t(x(t))$, for all $t\epsilon T$. However, it is <u>not</u> true that every imputation arises in the natural way from a maximal allocation. We define a <u>transferable utility competitive equilibrium tuce</u> for $(T,V,E,[U_t]_{t\epsilon T})$ as an allocation $[x(t)]_{t\epsilon T}$ and price vector $p \epsilon R^\ell$ such that $x(t)$ solves $\max_{y \geq 0} U_t(y) - p^t(y - I(t))$ for all $t\epsilon T$.* If in addition

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*Observe that if $[x(t)]_{t\ T}$ is a tuce, then $x(t)$ is maximal for T. Note that this is the same definition given before for $\bar{E}$.

$p^t(x(t) - I(t)) = 0$ for all $t \in T$, we call $[x(t)]_{t \in T}$ a natural transferable utility competitive equilibrium (ntuce). A competitive payoff configuration is given by a payoff configuration $[x_t]_{t \in T}$ such that $x_t = U_t(x(t)) - p^t(x(t) - I(t))$ for all $t \in T$ where $[x(t)]_{t \in T}$ is a tuce for $(T,V,E,[U_t]_{t \in T})$. If $[x(t)]_{t \in T}$ is a ntuce, we call $[x_t]_{t \in T}$ a natural competitive payoff configuration. Note that several tuce can give rise to the same competitive payoff configuration. We investigate the properties of a competitive payoff configuration and the relationship between the pareto optimal allocations and competitive equilibria in E and the imputations and tuce in $(T,V,E,[U_t]_{t \in T})$ more precisely in the following theorem.

Let us assume that the economy $E = (t, I(t), >_t)$ and the representing family of utilities $[U_t]_{t \in T}$ in the game $(T,V,E,[U_t]_{t \in T})$ satisfy the following properties:

(1) $|T|$ is finite

(2) $I(t) >> 0$, that is each trader owns a strictly positive quantity of each good. Since $|T|$ is finite we can find strictly positive vectors a and b such that $a \leqq I(t) \leqq b$ for all $t \in T$.

(3) The utilities $[U_t]_{t \in T} = \upsilon$ are all

(a) continuously differentiable (C') functions on R^{ℓ}_{+}

(b) and monotonic, so that the gradients ∇U_t are strictly positive. Again since $|T|$ is finite and the ∇U_t are continuous, the ∇U_t are uniformly positive, that is for any compact set $F \subset R^{\ell}_{+}$ there exist vectors $\bar{a}$ and $\bar{b} \in R^{\ell}_{+}$ such that $0 << \bar{a} \leqq U_t(x) \leqq \bar{b}$ for all $t \in T$ and $x \in F$.

For finite games we shall often require that the utilities are

(c) concave, that is for any x and $y \in R^{\ell}_{+}$,

$U_t(x) \leqq U_t(y) + \nabla U_t(y)^t(x-y)$ for all $t \in T$.

(d) strictly concave: $U_t(x) < U_t(y) + \nabla U_t(y)(x - y)$ for $x \neq y$.

When we deal with infinite economies we shall replace assumptions 3c,d with the requirement that the utilities be uniformly bounded; there exists on M such that $U_t(x) \leqq M$ for all $t \in T$ and $x \in R^{\ell}_{+}$. In that case the utilities U_t are not everywhere dominated by a linear function, as in the case of concave utilities, but they cannot exceed any linear function for long.

<u>Theorem</u>: If $E = (t, I(t), >_t)$ is an economy with a representing family of utilities $[\bar{U}_t]_{t\in T}$ satisfying assumptions 1,2,3a,b,c,d then the following relations hold between E and any derived game $(T,V,E,[U_t]_{t\in T})$ satisfying assumptions 1,2,3a,b,c,d:

(1) For every game $(T,V,E,[U_t]_{t\in T})$ there exists a unique maximal allocation $[x(t)]_{t\in T}$ for T which gives rise to a vector of Lagrange multipliers $p >> 0$. Moreover the pair $[x(t)]_{t\in T}$, p is the unique tuce for the game $(T,V,E,[U_t]_{t\in T})$. Therefore there is a unique competitive payoff configuration $[z_t]_{t\in T}$ for $(T,V,E,[U_t]_{t\in T})$ given by $z_t = U_t(x(t)) - p^t(x(t) - I(t))$ for all $t \in T$. The competitive payoff configuration $[z_t]_{t\in T}$ is in the core of $(T,V,E,[U_t]_{t\in T})$; in particular it is an imputation. The maximal allocation $[x(t)]_{t\in T}$, which also give rise to an imputation $[x_t]_{t\in T}$, $x_t = U_t(x(t))$ in the natural way, is pareto optimal in E. Thus every game $(T,V,E,[U_t]_{t\in T})$ singles out a single pareto optimal allocation $[x(t)]_{t\in T}$ in E which in turn gives rise to two imputations $[z_t]_{t\in T}$ and $[x_t]_{t\in T}$ back in $(T,V,E,[U_t]_{t\in T})$. The two imputations $[z_t]_{t\in T}$ and $[x_t]_{t\in T}$ are identical if and only if $[x(t)]_{t\in T}$ is a competitive equilibrium for E and a ntuce for $(T,V,E,[U_t]_{t\in T})$.

(2) To every pareto optimal allocation $[x(t)]_{t\in T}$ in E, we can find a game $(T,V,E,[U_t]_{t\in T})$ for which $[x(t)]_{t\in T}$ is the unique maximal allocation for T. In particular, if $[x(t)]_{t\in T}$ is a competitive equilibrium in E,

then we can find a game $(T,V,E,[U_t]_{t\epsilon T})$ in which $[x(t)]_{t\epsilon T}$ gives rise in the natural way to a competitive payoff configuration. Schematically the theorem states:

$(T,V^1,E,[U_t^1])$

$[x^1(t)]_{t\epsilon T}$ is maximal and is the tuce

$[x_t^1]_{t\epsilon T} = [U_t^1(x^1(t)]_{t\epsilon T}$ is an imputation

$[z_t^1]_{t\epsilon T} = [U_t^1(x^1(t) - p^t(x^1(t) - I(t))]_{t\epsilon T}$ is the competitive payoff configuration.

$E = (t, I(t), >_t)_{t\ T}$

pareto optimal allocations:

$[x^1(t)]_{t\epsilon T}$

$[x^2(t)]_{t\epsilon T}$

$[x^3(t)]_{t\epsilon T}$

$(T,V^2,E,[U_t^2]_{t\epsilon T})$

$[x^2(t)]_{t\epsilon T}$ is maximal and is the tuce

$[x_t^2]_{t\epsilon T} = [U_t^2(x^2(t)]_{t\epsilon T}$ is an imputation

$[z_t^2]_{t\epsilon T} = [U_t^2(x^2(t) - p^t[x^2(t) - I(t))]_{t\epsilon T}$ is the competitive payoff configuration.

Suppose $[x^3(t)]_{t\epsilon T}$ is a competitive equilibrium in E

$(T,V^3,E,[U_t^3]_{t\epsilon T})$

$[x^3(t)]_{t\epsilon T}$ is maximal and is the tuce

$[x_t^3]_{t\epsilon T} = [z_t^3]_{t\epsilon T} = [U_t^3(x^3(t)]_{t\epsilon T}$ is the competitive payoff configuration.

Since the theorem is well known, although proofs of its various parts are scattered throughout the literature, we have confined our leisurely proof to the appendix. The main result is that the ntuce and the competitive equilibria in E are the same, and the ntu pareto optima and the pare to optimal allocation in E are the same (by ntupo we mean any allocation $[x(t)]_{t\epsilon T}$ in E for which there is a game $(T,V,E,[U_t]_{t\epsilon T})$ in which $[x(t)]_{t\epsilon T}$ gives rise to an imputation $[x_t]_{t\epsilon T}$, $x_t = U_t(x(t)$ in the natural

way). Similarly, we can show that the ntucore in E is contained in the core of E ; it is only true for large economies however that the ntucore is the same as the core of E .

Now we are ready to define the value of a game and the value allocations in E . Suppose we have a game (T,V) , $T = [1,\ldots,N]$. Let us order the players at random, i.e. in one of the $N!$ possible ways. Consider player j , and let S be the set of all those players who come before j in the randomly selected ordering 0 . Look at $\phi_j^0 \equiv V(S\cup[j]) - V(S)$, where 0 refers to the ordering chosen. Note that $\sum_{j=1}^{N} \phi_j^0 = [V(1)-V(\phi)] + [V(1,2]-V(1)] + [V(1,2,3)-V(1,2)] + \ldots + [V(T)-V(T^{\sim}(N))] = V(T)$, assuming $V(\phi) = 0$. Define the value of player j in the game (T,V) as $\phi_j \equiv \frac{1}{N!} \sum_0 \phi_j^0$ where 0 varies over all possible orderings. Then still $\sum_{j\in T} \phi_j = V(T)$, so the value payoff is an imputation.*

The value was invented by Lloyd Shapley in 1953 and was an attempt to theorize how much a rational player would pay to play in a given game. It has several convenient properties, for instance $\phi_j(V) + \phi_j(V') = \phi_j(V+V')$ etc. It is also an attempt to suggest what a fair payoff would be.

Now consider our economy E and a corresponding game $(T,V,E[U_t]_{t\in T})$. For that particular game we can define the value payoff $[\phi_t(V)]_{t\in T}$. It is crucial to realize that a different representing family $[U_t]_{t\in T}$ of utilities will give rise to a different value payoff. If $[x(t)]_{t\in T}$ is an allocation for E and if there is a game $(T,V,E,[U_t]_{t\in T})$ derived from E such that $\phi_t(V) = U_t(x(t))$ for all $t\in T$, then we call $[x(t)]_{t\in T}$ a value allocation for E .

Thus in a game (T,V) , whether it is derivable from an economic market or not, the value always exists and is always an imputation, and is always

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* $\phi_j = \sum_{n=0}^{N-1} \frac{1}{N} \sum_{\substack{|S|=n \\ i\notin S}} \frac{1}{\binom{N-1}{n}} (V(S\cup[j]) - V(S))$ is an equivalent definition.

unique. Note that the notion of value allocation in E depends on a transferable utility game representation so that by definition the value allocations in E and what we might call the ntu value allocations are exactly the same.

The Bargaining Set was invented by Aumann and Maschler in 1963 in an effort to capture the payoffs that could occur in a game (T,V) when agents actually bargained with each other, weighing the desire for higher payoffs against the risk that excessive greed would bring reprisals from the other traders. To facilitate their thinking Aumann and Maschler reported that they gathered groups of friends and put to them the problem of dividing the pie. As a result of their experiments, Aumann and Maschler proposed several different definitions, but the one most convenient for us and one often used in the literature is based on the following intuition: A payoff $[x_t]$ is stable iff whenever a trader i sees a way of forming a coalition S that promises to pay more to each member than $[x_t]$ did, he neglects to "start the revolution" for fear that some trader j, alarmed by the formation of S, will form a counterobjecting coalition $\begin{matrix} R \ni j \\ R \not\ni i \end{matrix}$ that guarantees j his x_j and also any of the members of S used in R whatever i had offered them, thus potentially leaving i alone and unable to get even the x_i he began with. Precisely, given an imputation $[x_t]_{t\epsilon T}$, an objection $(S,[y_t]_{t\epsilon S},i_0)$ is a coalition S, a feasible payoff for S, $\sum_{t\epsilon S} y_t \leq V(S)$, and a leader $i_0 \epsilon S$ such that $y_t > x_t$ for all $t\epsilon S$. A counterobjection to $(S,[y_t]_{t\epsilon S},i_0)$ is any $(R,[z_t]_{t\epsilon R})$ $R \neq \emptyset$, $R \not\ni i$, and such that $\sum_{t\epsilon R} z_t \leq R$ and $z_t \geq x_t$ if $t\epsilon R$ and $z_t \geq y_t$ if $t\epsilon R \cap S$. The Bargaining Set is the set of imputations $[x_t]$

such that for any objection to $[x_t]$ there is a counterobjection.*

Example: Let $T = [1,2,3]$, $V[1] = V[2] = V[3] = 0$; $V[1,2] = V[2,1] = V[1,3] = V[1,2,3] = 1$ (This is a voting game, majority wins). Then the core is empty but (1/3,1/3,1/3) is in the Bargaining Set. It so happens that (1/3,1/3,1/3) is also the value payoff configuration.

Theorem 2: For any game (T,V) the Bargaining Set is nonempty.

Proof: Peleg, 1964.

For any game (T,V) the Bargaining Set contains the core. Thus if the core is nonempty, which in a game $(T,V,E,[U_t]_{t\epsilon T})$ satisfying assumptions 1-3a,b,c,d it must be since it contains the competitive payoff configuration, the Bargaining Set is trivially nonempty. If the utilities are not concave, there may be no tuce and no payoff configuration in the core. But from Theorem 2 we know that the Bargaining Set will still be nonempty.

If we were to be consistent with the rest of this section, we would now define the ntu Bargaining Set in E . However, we shall refrain from doing so, putting off a discussion of an ordinal Bargaining Set until Section 10. We remark now that it is not at all obvious for an arbitrary economy E satisfying assumptions 1,2,3a,b that the ntu Bargaining Set is always nonempty.

The Bargaining Set for a game (T,V) is an ingenious idea (and one that according to Theorem 2 is always meaningful) which formalizes our intuitive notion of how traders weigh the desire for further profit against the risk of grave loss in deciding what deals to strike up (and whether to upset the status quo). Nevertheless we shall change the Bargaining Set in this section in a way

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*Actually the definition we have given is already a slightly revised version of the original Aumann, Maschler definition. They envisaged a situation in which a trader i , unhappy with the way $V(T)$ was divided by $[x_t]_{t\epsilon T}$, would object against a specific trader $j\notin S$, who would in turn have to lead (be a part of) the counterobjection R . Out definition, which was suggested by Shubik and Shapley in an unpublished paper and used in Shapley's convergence paper, makes it easier to counterobject and therefore enlarges the Bargaining Set.

which tends to make it bigger. Since ultimately we are going to prove that even after it is enlarged, the Bargaining Set shrinks to the core as the number of traders in the economy T increases to infinity, this change does not affect the convergence of the original Bargaining Set.

We propose to drop the requirement that the Bargaining Set contain only imputations. It may well be that some methods of organizing society's natural resources do not produce pareto optimal payoff configurations (imputations) $[x_t]_{t \in T}$, yet these are still stable because the risks of attempting a re-organization are too great. For instance in our simple three person example, the payoff configuration (1/4,1/4,1/4) is not an imputation, but trader 1 may not be willing to lead an objection [1,2,3] promising everyone $x_t = 1/3$ for fear that traders 2 and 3, awakened by 1's activity to the fact that they could do better, might form the counterobjection [2,3] guaranteeing each other 1/2 and leaving trader 1 with nothing (i.e. less than the 1/4 he started with). Similarly it may be, using our production function interpretation of a transferable utility economy, that there are viable bargains which can be struck up between the various producers of food or energy which do not maximize the aggregate production of food or energy. Indeed one of the most interesting questions we can ask is, under what conditions will the Bargaining Set payoff configurations be imputations as well? Surprisingly, for large economies the Bargaining Set payoff configurations will nearly be imputations.

We formally set down our new definition of the Bargaining Set for finite games: The Bargaining Set BS of a finite game (T,V) consists of all these payoff configurations $[x_t]_{t \in T}$, $\sum_{t \in T} x_t \leq V(T)$ such that to every objection $(S,[y_t]_{t \in S}, t_0)$ satisfying:

(1) $t_0 \in S \subset T$

(2) $\sum_{t \in s} y_t \leq V(T)$

(3) $y_t > x_t$ for all $t \in S$

there exists a counterobjection $(R, [W_t]_{t \in R})$ satisfying

(1) $R \neq \emptyset$, $t_0 \notin R$

(2) $\sum_{t \in T} W_t \leq V(R)$

(3) $W_t \geq x_t$ if $t \in R \sim S$

$W_t \geq y_t$ if $t \in R \cap S$

Clearly this Bargaining Set contains the original Aumann-Maschler Bargaining Set, hence the Peleg existence theorem applies a fortiori to this as well. We shall prove in the next few sections that as the number of traders $|T|$ goes to infinity, the Bargaining Set converges to the core. Of course we shall have to find a suitable metric with which to measure the convergence, and also we shall have to explain how, as the number of traders goes to infinity so that by design each trader becomes insignificant, it can make sense to consider a set whose definition depends so crucially on an individual leader of an objection. In fact, we shall have to change the definition one more time.

SECTION III

Infinity and the Bargaining Set

We are interested in examining the consequences of the competitive hypothesis that in a market economy one agent, by his own actions, cannot affect the terms of trade (the prices) faced by himself or any other trader. In particular we shall show that in a transferable utility economic game $(T,V,E,[U_t]_{t \in T})$ satisfying the competitive hypothesis (were there a functioning market), the Bargaining Set and the Core are identical.

The first problem is to find a rigorous mathematical model which embodies the competitive hypothesis. Since in a market environment, the actions available to an agent amount essentially to determining how many of his own resources to offer to the "market" and which he will want in return, the competitive hypothesis implies a situation in which a single agent has under his exclusive control resources so modest that were he to remove almost all (or all) of them from the "market" he could not affect the terms of trade faced by any other agent in the economy. Yet this in turn implies that his resources as a proportion of the total available to the "market" must be negligible. It suggests immediately a notion of an infinite number of traders and a finite bound on the variation of the individual traders so that "at infinity" the significance of a single agent will be negligible.

It was first conjectured by Shubik, in the middle 60's after Scarf and Debreu had proved the convergence of the core to the competitive equilibria in a <u>replicated economy</u>, that <u>all</u> of the solution concepts defined in the last section are the same "as the number of traders goes to infinity." Of course one must have a model that can represent a "nearly" infinite number of traders and also allow a meaningful comparison between the various solution concepts

with a possibly changing number of traders. We can easily model the core as a subset of $R^{|T|}$ consisting of all vectors of the form $[x_t]_{t\varepsilon T}$, $x_t = U_t(x(t))$ for $[x(t)]_{t\varepsilon T}$ in the core of E and $[U_t]_{t\varepsilon T}$ any representing utility family. But as T grows, how can we say that a point in $R^{|T_1|}$ is closer to the competitive equilibria of E_1, then a point in $R^{|T_2|}$ is to the competitive equilibria of E_2?

Scarf and Debreu solved the problem elegantly by postulating a sequence of replicated economies. There were only n types of traders so that $E = E_1$ consisted of 1 trader of each type (where a type meant a preference $>_t$ and endowment $I(t)$), E_2 consisted of two traders of each type, etc. Scarf and Debreu also assumed strictly convex preferences, so that each trader of the same type must get exactly the same commodity bundle in any core allocation.[1] They then could compare the cores in E_K with those in E_j because they could be represented in the same Euclidean space. They proved in 1963 that

$$\bigcap_{K=1}^{\infty} \text{core } E_K = \text{competitive equilibria of } E .$$

Since their proof really was an equivalence in the limit anyway, it was not long before Aumann proposed a second model of an economy as a measure space of agents. Only in such a model, he argued, were the assumptions of perfect competition (price unaffected by the behavior of any single agent) justified in any case. And in the continuous model Aumann proved in 1964 the equivalence between the core and the competitive equilibria, without assuming convexity of preferences, or even transitivity.

In 1964 Shapley began in earnest to prove convergence of the various solution concepts in the derived game as opposed to in the original economic market. Thus he proved the convergence of the value payoff configuration to

[1] Or assume convex preferences, then each trader of the same type must get the same utility from the core allocation of commodity bundles.

the competitive payoff configuration in a sequence of replicated economic games (obviously two traders of the same type have the same value payoff) and a year earlier he and Shubik had shown in a two type case that the core shrinks to competitive payoff configuration. In 1973 he and Aumann proved the equivalence of the value payoff configuration and the competitive payoff configuration in a game derived from a measure space of agents. In 1975 Aumann invented the idea of a value allocation, and showed the equivalence of the competitive equilibria and value allocations in E^*. Essentially, since as we saw, every competitive allocation in E can be represented as a ntuce in some game $(T,V,E,[U_t]_{t\epsilon T})$ and every value allocation is by definition represented as a ntu value allocation in some game it follows at once that if the value payoff configuration and the competitive payoff configuration are always identical in every game, and if a ntuce is a competitive equilibria in E, then the competitive equilibria and value allocations are identical.

Meanwhile Don Brown, in collaboration with the inventor of nonstandard analysis Abraham Robinson, proposed the Nonstandard model of infinite economies. In 1970 Brown and Robinson proved the equivalence of the core and competitive equilibria for nonstandard E, and in 1976 Brown and Loeb proved the equivalence of the value and competitive payoff configurations in nonstandard economic games and also the equivalence of the competitive equilibria and value allocations for nonstandard E. A discussion of the merits of each of the alternative models of infinite economies is given later.

Finally in 1976 Shapley proved the convergence of the Bargaining Set to the Core in a sequence of replicated economic games.

It should be mentioned that another property, namely the existence of

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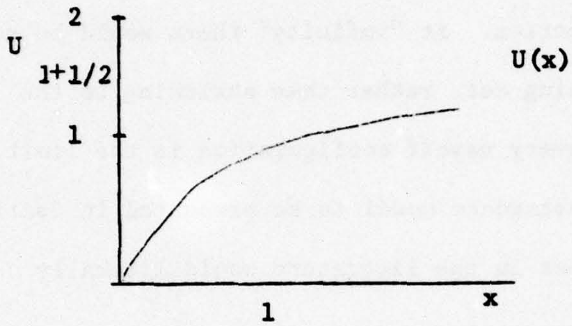
*Recall that E is an economy and $(T,V,E,[U_t]_{t\epsilon T})$ is a derived economic game.

equilibria (or approximate equilibria) with nonconvex preferences was proved in a simplified replicated economy by Shapley and Shubik in 1963, in the nonatomic model by Aumann in 1966, and in the nonstandard framework by Brown in 1972.

In summary we see that in all three models of perfect competition, the competitive payoff configurations, value payoff configuration, the core payoff configurations, and the Bargaining Set payoff configurations (should) coincide for each "infinite" game $(T,V,E,[U_t]_{t\varepsilon T})$ and the competitive allocations, the core allocations, the (ntu) value allocations, and the Bargaining Set allocations (should) coincide in "infinite" economies E. The only results which remain unproved are the Bargaining Set equivalence for nonatomic and nonstandard transferable utility games and economies. We shall postpone a discussion of Bargaining Set allocations for economies until Section 10.

The usual Bargaining Set equivalence for the nonatomic, measure theoretic model evidently cannot be proved until its definition is altered in such a way that it does not depend crucially on a single trader, who literally disappears in the nonatomic model. For a while it was felt that it might be possible to prove the equivalence in the nonstandard model since there the notion of an individual trader is definable. Nevertheless the purpose of the nonstandard model is to make rigorous the competitive hypothesis that one trader does not matter. It would be paradoxical indeed if the standard definition of the Bargaining Set, with its crucial dependence on one trader could be shown to be the same as the competitive payoff distribution. We show easily in Section 6 that such an equivalence does not hold. In fact, we argue now that the standard definition of the Bargaining Set is not appropriate even for replicated sequences of finite games, the Shapley convergence notwithstanding.

Consider the following example, which satisfies all the assumptions of the Shapley convergence theorem. Let there be one good in the economy E and let the three traders in $E = E_1$ have utility functions $U_t(W) = W$, $t = 1,2,3$ and initial endowments $I(t) = 1$, $t = 1,2,3$. Then $V(T_1) = V[1,2,3] = 3$, and more generally $V(S) = |S|$ for all $S \subset T_1$. (We could easily make the utility functions satisfy all of our assumptions and make them strictly concave and uniformly bounded as well. The only point should be that $U(1) = 1$, so that $V(S) = |S|$ for all S.)



Consider also the payoff configuration giving nothing to the first trader, 1 utile to the second trader, and 2 utiles to the last trader, $x = (0,1,2)$. Then x is clearly feasible, and it is not in the core, which contains the single imputation $\bar{z} = (1,1,1)$.

We certainly want it to be true that if we replicate the economy E a sufficient number of times, we will eventually find a justified objection to the replicated payoff configuration x (that is an objection that has no counterobjection). In fact, even before the economy is replicated at all there is a justified objection $(S = [1,2], (y_1,y_2) = (1/2,3/2), t_0 = [1])$. Note that $y_t - x_t = 1/2$ for all $t \varepsilon S$ and that clearly there can be no counterobjection, since for every $t \neq t_0$, $y_t > 1$ if $t \varepsilon S$ and $x_t >. 1$ if $t \notin S$. But now see what happens when we replicate the economy, giving 2 traders of each type and letting $x = (0,0,1,1,2,2)$. Now a justified objection is

$(S = [1,2,3,4]$, $(y_1,y_2,y_3,y_4) = (1/4,5/4,5/4,5/4)$, $t_0 = [1])$. Note that $y_t - x_t = 1/4$ for all the traders of type 2 in S . Note that we are constrained in making our justified objection to giving each trader in S , except t_0 , strictly more than 1. Thus for the n^{th} replication, a justified objection will be $(S = [1,2,\ldots,2n]$, $(y_1,\ldots y_{2n}) = (1/2n,1+1/2n,\ldots,1+1/2n)$ and now for all traders of type 2, the best justified objection only results in a net gain of $1/2n$. Thus the more often we replicate the economy, the less compelling is the justified objection. At "infinity" there would be no objection at all. Indeed the Bargaining Set, rather than shrinking to the core, in a sense expands to include every payoff configuration in the limit. In the nonatomic model and in the nonstandard model to be presented in Section 6, the Bargaining Set as defined so far in the literature would literally contain every payoff configuration.

We are led then to a new definition of the Bargaining Set; one that is appropriate for large economies but at the same time conforms to the spirit of the original definition. Let us examine the intuition of the Bargaining Set a little more closely, bearing in mind the production economy interpretation of a transferable utility economy described in the last section. A given payoff configuration $[x_t]_{t\epsilon T}$, the abstract result of a host of contracts made between individual producers, is unacceptable to a trader t_0 if he can see a way to reorganize part of the production process in a way which will yield himself and the rest of his coalition S a greater payoff $[y_t]_{t\epsilon S}$. Of course this will involve the breaking up of many of the original contracts and the consequent risk of retaliation in the form of a counterobjection ; the Aumann-Maschler definition postulates that no objection can be made without a leader and that no trader will act as leader if there is the possibility of a counterobjection,

to which every collobarotor in the "revolution," except the leader himself, is susceptible to bribery. (R can be any set so long as $t_0 \notin R$.). This seems to us like good psychology: no major reorganizations take place without leadership and once a leader has identified himself with a cause he will often cling tenaciously to it even though there might be further profit if he abandoned it. To put it another way, a leader can be sure he will be loyal to his own cause.

In a small group, say of four or five agents, each individual may have special goals and there is no reason to count on the unswerving agreement of any of the others of the group in a dispute involving only members of the small group. On the other hand, as the group grows larger, so that we might imagine a small town, there will arise small coalitions, like the family, which in some kinds of disputes can count on each other never to be tempted by offers which may prove detrimental to others in the coalition (family). And as the society grows larger, reorganizations require more complicated decisions and possibly more leaders. We can formalize this intuition, and also eliminate the problem we have been discussing, by replacing the leader of the objection t_0 with a very small set of leaders K. For the nonatomic model, define the δ Bargaining Set as any payoff configuration x_t, $\int_T x_t d\mu \leq V(T)$ such that to every objection $(S,[y_t]_{t\in S},K)$ satisfying:

(1) $K \subset S \subset T$, K and S measurable, $\mu(S) > 0$

(2) $\mu(K) \leq \delta$

(3) $\int_S y_t d\mu = V(S)$

(4) $y_t > x_t$ a.e. in S

there exists a counterobjection $(R,[W_t]_{t\in R})$ satisfying:

(1) $R \subset T$, R measurable, $\mu(R) > 0$ and $R \cap K = \emptyset$

(2) $\int_R W_t d\mu \leq V(R)$

(3) $W_t \geq y_t$ a.e. on $S \cap R$

$W_t \geq x_t$ a.e. on $R \sim S$

Note that for $\delta_1 < \delta_2$, $\beta S_{\delta 1} \supset \beta S_{\delta_2}$ *. We define the Bargaining Set as the union of all the δ-Bargaining Sets for $\delta > 0$,

$$\beta S = \bigcup_{\delta > 0} \beta S_\delta .$$

We will give a similar definition of the nonstandard Bargaining Set. For sequences of finite economies, we define anagously the δ-Bargaining Set and then show that for any $\delta > 0$, the δ-Bargaining Set converges to the core.**

With this definition we are ready to complete the chain of equivalence theorems mentioned earlier. In the next section we give a loose, intuitive account of why all such payoff configurations should be expected to converge to each other and to the payoff configuration $[V(T) - V(T \sim [t])]_{t \epsilon T}$ as $|T| \to \infty$. We remark that once the nonstandard model is shown to preserve the competitive hypothesis, that the presence or absence of one trader does not significantly affect the equilibrium prices, all the equivalences follow almost effortlessly.

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*The smaller K is, the easier it is to counterobject and thus the larger is βS_δ.

**See Section 9. For the finite case everything is the same except that we replace $\mu(K) \leq \delta$ with $\left|\frac{K}{T}\right| \leq \delta$.

SECTION IV

The Essential Assumption, Pseudoconcavity, and The Law of Large Numbers

The intuition behind the proofs of all the foregoing equivalences can be outlined, at least in its most essential aspects, in a short space below.[1] All of the proofs basically depend on the same three ideas: the convexifying effect of many traders, the law of large numbers, and what I call the essential assumption that traders are more alike than they are different. This last assumption, no matter how it is disguised, is present in every model and in every equivalence proof. Indeed it must be, for it is the very heart of the perfect competition hypothesis. Thus Scarf-Debreu and Shapley explicitly assume types so that after numerous replications the ratio of types to traders becomes arbitrarily small. Aumann and Brown in their value equivalence papers assume bounded differentiable utility functions (among other things that means a uniform upper bound M on all the utility functions). In Section 6 we demonstrate that if a sequence of ever larger economies always draws its utilities from a bounded differentiable family $\mathcal{U}$ of utilities, then as the number of traders increases various types can be distinguished such that the ratio of the number of types (which is itself becoming infinite) to the number of agents in the economy nevertheless goes to zero. With nonstandard analysis we can rigourize the intuition that "in the limit" with bounded differentiable utilities we get almost a type economy with unequal replications (more of one type than another) where the number of types is infinite but the ratio of types to traders is "infinitesimally" small.

We say that a function $U(x)$ is concave at x when $U(y) \leq U(x) + \nabla U(x)(y-x)$ for all $y \in R^{\ell}$ and pseudoconcave if for some $p \in R^{\ell}$, $U(y) \leq U(x) + p^t(y-x)$ for all $y \in R^{\ell}_{+}$. A function $U(x)$ is concave if it is

[1] The intuition of this section is made rigorous in sections 6, 8, and 9.

concave at all x, Suppose that $x(t)$ is maximal for T in the game $(T,V,E,[U_t]_{t\epsilon T})$ where the utilities are <u>not</u> necessarily concave.[1] Then we must have that whenever $x_i(t) > 0$ and $x_i(s) > 0$, $\nabla_i U_t(x(t)) = \nabla_i U_S(x(s)) \equiv p_i$, for if $\nabla_i U_t(x(t)) > \nabla_i U_t(x(s))$, then by shifting a little of good i from trader s to trader t we could increase $\sum_{t\epsilon T} U_t(x(t))$, contradicting the maximality of $[x(t)]_{t\epsilon T}$. We can show, however, that if $|T|$ is very large, then it must nearly be the case that for all $y \epsilon R_+^\ell$, $U_t(y) \leq U_t(x(t)) + p^t(y-x(t))$, that is that $U_t(x)$ is pseudoconcave at $x = x(t)$. (In fact, if $x(t) >> 0$, $\nabla U_t(x(t)) = p$ and $U(x)$ is nearly concave at $x = x(t)$.) For suppose that $U_{t_0}(y) > U_{t_0}(x(t_0)) + p^t(y-x(t_0))$ for some $y\epsilon R_+^\ell$. Then we can give to trader t_0 the commodity bundle y instead of $x(t_0)$ resulting in a gain of $U_{t_0}(y) - U_{t_0}(x(t_0))$, while the decline in consumption $y - x(t_0)$ for the rest of T can be spread so thinly among traders who hold positive quantities of the commodities taken from them (so that $\nabla_i U_t(x(t)) = p_i$) that the change in $\sum_{t\neq t_0} U_t(x(t))$ is almost exactly $-p^t(y-x(t))$. But again this contradicts the maximality of $[x(t)]_{t\epsilon T}$. Hence for a maximal allocation $[x(t)]_{t\epsilon T}$ in a large economy T, we must have (approximately) for all $t\epsilon T$, $U(y) \leq U(x(t) + p^t(y-x(t))$ even if the utilities U_t are not concave.

The law of large numbers says that if we have a large collection of $|T|$ objects of v different types, where $v/|T|$ is very small, and if we randomly draw out $|S| < |T|$ elements, where $V/|S|$ is also small, then with probability nearly 1, the collection S will have nearly the same ratio of types as the original collection T.

In economic games $(T,V,E,[U_t]_{t\epsilon T})$ without a central planner and without a coordinating system of market prices, traders must bargain with and threaten

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[1] That is, suppose $V(S) = \sum_{t\epsilon S} U_t(x(t)) = \text{Max} \sum_{t\epsilon S} U_t(y(t))$ s.t. $y(t)\epsilon R_+^\ell$ and $\sum_{t\epsilon S} y(t) = \sum_{t\epsilon S} I(t)$. We are demonstrating that any allocation $[x(t)]_{t\epsilon S}$ gives rise to prices p and that if S is large, U_t is pseudoconcave at $x(t)$ with respect to p for all $t\epsilon T$.

other traders to gain the maximum possible payoff x_t. The only direct bargaining power a trader has, and the only immediate threat he can make, is to quit the economy removing his resources from general use, thus causing a total loss of $V(T) - V(T\sim[t])$. Each trader t would like to demand at least the payoff $V(T) - V(T\sim[t])$; if he doesn't get it he can threaten to quit the economy. The difficulty is that for finite games in general (always if the utilities are concave) it is impossible to give each trader the payoff $V(T) - V(T\sim[t])$, that is $\sum_{t\in T} [V(T) - V(T\sim[t])] > V(T)$. Hence some simplifying principle of allocation, whether it is based on power (core) or fairness (value payoff configurations), or some notion of what makes a viable contract (Bargaining Set) must be invoked to achieve a stable payoff configuration in an apparently chaotic situation. All of these principles essentially reduce to the payoff configuration $[V(T) - V(T\sim[t])]_{t\in T}$ when it is feasible and when the hypothesis of perfect competition holds (of course this must be shown in each case). The riddle of why all the solution concepts approach each other in the limit is mainly the riddle of why $\sum_{t\in T} [V(T\sim[t])]$ becomes nearly feasible as $|T|$ gets very large. And this we can explain.

Observe that if we add traders one by one to the economy, and give them each their marginal contribution, the resulting payoff contribution is feasible. That is, if we take any (random) ordering of the traders, $t_1, t_2, \ldots t_{|T|}$, then

$$\sum_{K=1}^{|T|} [V([t_1,\ldots t_K]) - V([t,\ldots t_{K-1}])] = [V([t_1]) - V(\emptyset)] + [V([t_1,t_2]) - V([t_1])]$$
$$+ \ldots + V(S) - V(S\sim[t_K]) + \ldots + V(T) - V(T\sim[t_{|T|}]) = V(T).$$

But if the number of traders in $|T|$ is large, and if we have made the essential assumption, then almost all of the coalitions $S = [t_1,\ldots,t_K]$ will have nearly the same mix of types as T itself. Hence for all such S, and therefore for almost all $t_K \in T$, it follows from th[illegible]seudoconcavity of $U_t(x)$ at $x = x(t)$ that the

marginal contribution of t_K to S, $V(S) - V(S\sim[t_K])$, will be almost the same as the marginal contribution of t_K to T, $V(T) - V(T\sim[t_K])$. To see this, observe that if we start from same large coalition S and a maximal allocation $[x(t)]_{t\in S}$ for S, and we replicate S to get a coalition $\bar{S}$ (so $\bar{S}$ contains exactly two copies of every trader in S), then the allocation $[x(t)]_{t\in\bar{S}}$ obtained by replicating $[x(t)]_{t\in S}$ must be maximal for $\bar{S}$, so that $V(\bar{S}) = 2V(S)$.* For suppose $[y(t)]_{t\in\bar{S}}$ is an allocation for $\bar{S}$, that is $\sum_{t\in\bar{S}} y(t) = \sum_{t\in\bar{S}} I(t) = \sum_{t\in\bar{S}} x(t)$. Then for any $\bar{t}\in\bar{S}$, $\bar{t}$ is a copy of some t in S, so $U_{\bar{t}}(y(\bar{t})) \leq U_{\bar{t}}(x(\bar{t})) + p^t(y-x(\bar{t}))$.[1] Hence $\sum_{\bar{t}\in\bar{S}} U_{\bar{t}}(y(\bar{t})) \leq \sum_{\bar{t}\in\bar{S}}[U_{\bar{t}}(x(\bar{t})) + p^t(y(\bar{t}) - x(\bar{t}))] = \sum_{\bar{t}\in\bar{S}} U_{\bar{t}}(x(\bar{t})) + p^t(\sum_{t\in\bar{S}} y(\bar{t}) - \sum_{t\in\bar{S}} x(\bar{t})) = \sum_{\bar{t}\in\bar{S}} U_{\bar{t}}(x(\bar{t}))$. Thus V is, heuristically speaking, a homogeneous function of degree 1; the first partial derivatives of V are therefore homogeneous of degree 0, intuitively speaking, and so it is reasonable to conclude that if S is a scaled down replica of T, then $V(S) - V(S\sim[t])$ $= V(T) - V(T\sim[t])$. But the law of large numbers assures us that S is almost a scaled down version of T for nearly all S in the chain $\sum_{K=1}^{|T|}[V([t_1,\ldots,t_K]) - V([t,\ldots t_{K-1}])] = V(T)$, so we must have $\sum_{t\in T}[V(T) - V(T\sim[t])] = V(T)$ as well.**

The value payoff for trader t in the game $(T,V,E,[U_t]_{t\in T})$ is defined as the simple average of trader t's marginal contribution in each of the $|T|!$ different orderings $t_1\ldots,t_{|T|}$. For very large $|T|$, as we just saw, the marginal contribution of t in nearly every ordering is almost equal to $V(T) - V(T\sim[t])$. Thus not only is the value payoff configuration $[y_t]_{t\in T}$

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*We assume that $|S|$ is also large, so that the $U_t(x)$ are pseudoconcave at $x = x(t)$ for $t\in S$.

**This intuition is made rigorous in Section VIII.

[1] We use the pseudoconcavity of $U_{\bar{t}}$ at $x(\bar{t})$ which holds since S is large and $[x(t)]_{t\in S}$ is maximal for S.

nearly the same as $[V(T) - V(T\sim[t])]_{t\epsilon T}$ for large $|T|$, but the semi-values obtained by using arbitrary weights summing to one for the marginal contributions in the $|T|!$ different orderings must also converge to the single payoff $[V(T) - V(T\sim[t])]_{t\epsilon T}$. Observe that for small economies T there is no reason for the semi values to even be imputations.[1]

The core payoff configurations of a transferable utility economic game are also easily seen to converge to the unique payoff configuration $[V(T) - V(T\sim[t])]_{t\epsilon T}$. If $[x_t]_{t\epsilon T}$ is in the core, then by definition $\sum_{t\epsilon T} x_t = V(T)$ and for any $t_0 \epsilon T$, $\sum_{t\epsilon T\sim[t_0]} x_t \geq V(T\sim[t_0])$. Hence $\sum_{t\epsilon T} x_t - \sum_{t\epsilon T\sim[t_0]} x_t \leq V(T) - V(T\sim[t_0])$, so for all $t_0 \epsilon T$, $x_{t_0} \leq V(T) - V(T\sim[t_0])$. As $|T|$ gets large, $V(T) = \sum_{t_0 \epsilon T} x_{t_0} \leq \sum_{t_1 \epsilon T} [V(T) - (T\sim[t_0])] \approx V(T)$, so $x_{t_0} \approx V(T) - V(T\sim[t_0])$.

Needless to say, in the limit the competitive payoff configuration $[x_t]_{t\epsilon T} = [U_t(x(t)) - p^t(x(t) - I(t))]_{t\epsilon T}$ also gives each trader t his marginal contribution $V(T) - V(T\sim[t])$. The intuition is similar to the previous cases: given a maximal $[x(t)]_{t\epsilon T}$ for T and any $t_0 \epsilon T$, if $|T|$ is big enough, the loss in total utility if t_0 leaves T can be no more harmful than the loss $U_{t_0}(x(t_0))$ added to the total loss of utility from spreading around the loss in consumption $I(t_0) - x(t_0)$ which must occur due to the removal of t_0's endowment. If $|T|$ is very large, that will be precisely $p^t(I(t_0) - x(t_0))$, hence for $|T|$ very big, $V(T) - V(T\sim[t_0]) \leq U_{t_o}(x(t_0)) + p^t(I(t_0) - x(t_0)) = U_{t_o}(x(t_0)) - p^t(x(t_0) - I(t_0)) = x_{t_0}$.

Finally the Bargaining Set payoff configurations, which like the semi-values but unlike any of the other solution concepts are <u>not</u> assumed to be imputations, must also converge to the single imputation $[V(T) - V(T\sim[t])]_{t\epsilon T}$.

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[1] Since the weights $(W_1^t \ldots, W_{|T|!}^t)$ can be different for different traders t .

We can show this most easily by showing that all the Bargaining Set payoff configuration, $[x_t]_{t\epsilon T}$ must nearly be in the core for $|T|$ very large. Suppose that $\max_S \frac{1}{|T|}(V(S) - \sum_{t\epsilon S} x_t)$ is greater than zero, and let the maximum be attained at S_0. Then for all $t\epsilon S_0$, $V(S_0) - V(S_0 \sim [t]) \geqq x_t$ (otherwise $S_0 \sim [t]$ would produce a greater "excess") and for all $t \notin S_0$, $V(S_0 \cup [t]) - V(S_0) \leqq x_t$ (otherwise $S_0 \cup [t]$ would produce a greater excess.) If S_0 is a nonnegligible part of T (which it must be if $\frac{1}{|T|}(V(S_0) - \sum_{t\epsilon S_0} x_t)$ is to be significantly greater than zero) and if $|T|$ is large, then $|S_0|$ is also large. Consider the payoffs $y_t = V(S) - V(S \sim [t])$ for all $t\epsilon S_0$. If $|S_0|$ is large, then we must have $\sum_{t\epsilon S_0} y_t = \sum_{t\epsilon S_0} V(S_0) - V(S_0 \sim [t]) = V(S_0)$. By changing the $[y_t]_{t\epsilon S}$ a little, reducing it for some t (the "leaders") and increasing it for others, we can transform $[y_t]_{t\epsilon S}$ into an objection. That there can be no counterobjection follows easily from the pseudoconcavity of $U_t(x)$ at $x = x(t)$ if $[x(t)]_{t\epsilon S_0}$ is maximal for S: if $R \subset T$, then

$$V(R) = \sum_{t\epsilon R \cap S_0} (V(S_0) - V(S_0 \sim [t]) + \sum_{t\epsilon R \sim S_0} (V(S_0 \cup [t]) - V(S_0)) < \sum_{t\epsilon R \cap S_0} y_t + \sum_{t\epsilon R \sim S_0} x_t .$$

We prove this rigorously in Section 8, and we also confirm there the rest of the preceding intuitive arguments.

Our intuition so far has been applied only to games $(T,V,E,[U_t]_{t\epsilon T})$ derived from economies E, but we can explain the core equivalence for a nontransferable economy E in the same fashion. Suppose $|T|$ is very large, and $[x(t)]_{t\epsilon T}$ is an allocation in the core of $E = (t, I(t), >_t)_{t\epsilon T}$, where we assume the $>_t$ are monotonic but not necessarily convex or even transitive or continuous. We would like to be able to find prices $p >> 0$ such that $p^t(x(t) - I(t)) = 0$ for almost all $t\epsilon T$ and such that if $y(t) >_t x(t)$, then $p^t(y(t) - I(t)) > 0$ for almost all $t\epsilon T$. So it is natural to consider the sets

$G_t = [z \epsilon R^\ell \,|\, z = y - I(t)$ and $y >_t x(t)]$ as Scarf-Debreu did in 1963 and Aumann did in 1964. Observe that for no subset A of T can it be true that $\sum_{t \epsilon A} g_t = z << 0$, for then $(A, [I(t) + g_t]_{t \epsilon A})$ would block $[x(t)]_{t \epsilon T}$. If we take $G = \bigcup_{t \epsilon T} G_t$, we need to find prices $p >> 0$ such that $p^t g > 0$ for all $g \epsilon G$. If we let $\hat{G}$ = convex hull of G, then it will suffice to find $p >> 0$ such that $p^t g > 0$ for all $g \epsilon \hat{G}$. Moreover if we could show that for large $|T|$, the convex sets R^ℓ_- and $\hat{G}$ are disjoint except for 0, then it would follow from the separating hyperplane theorem that there is a $p >> 0$ such that $p^t g \geqq 0$ for all $g \epsilon \hat{G}$. Suppose that for some $z << 0$, $z \epsilon \hat{G}$, that is $z = \sum_{i=1}^{\ell+1} d_i g_{t_i}$ where $g_{t_i} \epsilon G_{t_i}$, $d_i \geqq 0$ for $i = 1, \ell+1$ (recall that by Caratheodary's theorem, any element of the convex hull of a set G contained in R^ℓ can be expressed as a linear combination of $\ell + 1$ vectors in G) and $\sum_{i=1}^{\ell+1} d_i = 1$. If the d_i were all equal to 1, so that we had $\sum_{i=1}^{\ell+1} g_{t_i} = z << 0$, we would have a contradiction since the coalition $A = [t_1, \ldots, t_{\ell+1}]$ would block $[x(t)]_{t \epsilon T}$, hence we could conclude that $\hat{G} \cap R^\ell_- = [0]$ and we would nearly be done.

Scarf-Debreu and Aumann in different ways invoke the familiar traders are more alike than different assumption to show that in effect we can take the $d_i = 1$. Since $\sum_{i=1}^{\ell+1} d_i g_{t_i} = z << 0$, by taking rational vectors $\hat{g}_{t_i} >> g_{t_i}$ and rational $\hat{d}_i$ sufficiently close to g_{t_i} and d_i respectively, we still have that $\sum_{i=1}^{\ell+1} \hat{d}_i \hat{g}_{t_i} = \hat{z} << 0$ and $I(t_i) + \hat{g}_{t_i} >_{t_i} x(t_i)$ for $i = 1 \ldots, \ell+1$. We can express each $\hat{d}_i$ as the ratio of two integers, N_i/M where M is a common denominator for the rationals $\hat{d}_i$. Then $\sum_{i=1}^{\ell+1} \frac{N_i}{M} \hat{g}_{t_i} = \hat{z} << 0$ and multiplying by M, $\sum_{i=1}^{\ell+1} N_i \hat{g}_{t_i} << 0$. Scarf-Debreu assumes a finite type economy in which

a core allocation must give each trader of the same type the same commodity bundle (this is the case if preferences are strictly convex). Then for $|T|$ very big, the set of traders of the same type as t_i all prefer $\hat{g}_{t_i}$ to what they were given by $[x(t)]_{t\varepsilon T}$ and there are more than N_i of them, hence there is an $A \subset T$ such that $\sum_{t\varepsilon A} g_t = \sum_{i=1}^{\ell+1} N_i \hat{g}_{t_i} << 0$.

Aumann, on the other hand, doesn't need to explicitly assume types but by using a model with an uncountable number of agents and assuming continuous preferences (so that the preferences are determined by their ordering of the countable rationals) he gets them anyway. More precisely, for every rational vector $r\varepsilon R^{\ell}$, consider the set $A_r \subset T$ of traders who prefer $r + I(t)$ to $x(t)$. If the Lebesgue measure of A_r is 0, exclude all the traders A_r from T . Since there are only a countable number of rational vectors r in R^{ℓ} , we have excluded at most a set of traders $S = \bigcup_{r\varepsilon Q} A_r$ of measure 0 . But for the traders that remain, $\hat{T} = T \sim S$, if $r\varepsilon R^{\ell}$ is rational and $r+I(t) >_t x(t)$ for some $t\varepsilon T$, then there is a set of traders of positive measure A such that $r + I(t) >_t x(t)$ for all $t\varepsilon A$. In particular for the rational vectors $\hat{g}_{t_i}$, we can find sets $A_i, \mu(A_i) > 0$, such that $\hat{g}_{t_i} \varepsilon G_t$ for all $t\varepsilon A_i$, and $i = 1\ldots,\ell$. All that remains is to take disjoint subsets $\hat{A}_i \subset A_i$ such that $\mu(A_i)/\mu(A_j) = \hat{d}_i/\hat{d}_j$ for all $i,j = 1\ldots,\ell+1$. Then $\int_{\hat{A}_1} g_t\, dt + \ldots + \int_{\hat{A}_{\ell+1}} g_{t_{\ell+1}}\, dt = 0$, hence $\hat{A} = \hat{A}_1 \cup \ldots \cup \hat{A}_{\ell+1}$ blocks $[x(t)]_{t\,T}$, and we can conclude that it was impossible that $z\varepsilon G \cap R^{\ell}_{-}$.

Thus in both proofs we get a price vector p such that $p^t g \geq 0$ for all $g\varepsilon G$ (where for Aumann $G = \bigcup_{t\varepsilon T\sim S} G_t$.). Hence if $y(t) >_t x(t)$, then $p^t y(t) \geq p^t I(t)$. At last both proofs need continuity of preferences to show that in fact $p^t g(t) > p^t I(t)$. If $y(t) >_t x(t)$, then for very small $\varepsilon > 0$,

$y(t) - \varepsilon e >_t x(t)$ where $e = (1,\ldots,1)^t$. Hence $p^t y(t) > p^t(y(t)-\varepsilon e) = p^t I(t)$ and the proof is complete.

Observe that nowhere in either the Debreu-Scarf or Aumann proofs did we use the fact that the consumption vectors $x(t)$ had to be in R_+^ℓ . Hence the same proofs show that in a transferable utility economy (where money can be held in negative quantities) with many traders $\bar{E}$, the core and competitive equilibria also coincide.*

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*Recently Bob Anderson has given an even simpler proof of the core equivalence which relies heavily on the assumptions that the consumptions sets are bounded from below $(x \geq 0)$ and that the initial endowments are uniformly bounded, but makes no assumptions other than monetonicity on the preferences.

SECTION V

An Introduction to Nonstandard Analysis

It is surprising that until now there has apparently been no short introduction to nonstandard analysis for the economist who has nc prior knowledge of advanced logic. Yet an introduction which would enable the economist to grasp the fundamental ideas of this new branch of mathematics, at least insofar as they apply to economics, does not seem impossible. Although there can be no true substitute for a close reading of a good text,[1] we attempt to set down in this essentially self-contained introduction (only one famous theorem from logic is used without proof) all of the ideas which need to be acquired to make a critical reading of the rest of this paper and some of the other works in the literature on nonstandard analysis as well. Along the way we try to make clear why the nonstandard model is so perfectly compatible with the notion of pure competition.

We suggested earlier that the fundamental hypothesis of pure competition - that no agent acting on his own could alter the terms which governed his transactions - implied a situation in which a single agent has under his exclusive control resources so modest that were he to remove almost all (or all) of them from the "market" he could not affect the terms of trade faced by any other agent in the economy. Yet this in turn implies that his resources as a proportion of the total available to the "market" must be negligible; so negligible that every agent ignores the change in action one agent undertakes and the market signals (i.e. prices) remain unaffected by those solitary actions. Thus Aumann [1] suggested as a model a measure space of agents; a

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[1] For instance, Abraham Robinson Nonstandard Analysis.

[1] Aumann 1964.

a single trader, thought of as a point t on the real line, has measure 0 and so his presence or absence makes literally no difference to the total resources available to the market. However, it is difficult to explain in such a model in what sense a trader even exists; recently Aumann has begun to identify a trader with an infinitesimal interval dt on the line.[1]

A similar paradox, arising from Leibnitz' creation of calculus, was exposed in Bishop Berkeley's brilliant article, written in the 18th century; "To an Infidel Mathematician." The Leibnitz method of differentiation, Berkeley argued, follows roughly the following pattern:[2] to differentiate $y = x^2$, set

(1) $dy = (x + dx)^2 - x^2 = x^2 + 2xdx + dx^2 - x^2 = 2xdx + dx^2$

then divide by dx, getting

(2) $\frac{dy}{dx} = 2x + dx$

and setting $dx = 0$, get

(3) $\frac{dy}{dx} = 2x$.

Berkeley reasoned that either $dx = 0$, in which case it is impossible to divide by dx in line 2, or else $dx \neq 0$, in which case it is contradictory to assume $dx = 0$ in line 3. The good bishop concluded by wondering "Hath mathematics its mysteries too?"[3]

Of course Leibnitz' argument can be repaired with the standard ε, δ limit arguments of modern mathematics. But there is another method, which also has the advantage of freeing the economic model of perfect competition

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[1] Aumann 1976. Of course this second formulation also has difficulties, one of which is that traders overlap.

[2] I am grateful to Professor Hilary Putnam for discussing this example.

[3] Bell's handbook of mathematics

from its apparent paradox. Suppose there is a way to enlarge the real number system to include "infinitesimals," exactly as there is a way to enlarge the real number system to include complex numbers, so that these "infinitesimals" are smaller than any "standard" real number, but different from zero. Then letting dx be an "infinitesimal," different from zero, we can indeed divide by dx in line 2 of Leibnitz' argument and conclude in line 3 that $\frac{dy}{dx}$ is "infinitesimally" close to $2x$. Similarly if one trader's endowment is only an "infinitesimal" fraction of the total, and if the agents are indifferent, even though they perceive the difference, between allocations that differ "infinitesimally", then one trader can have no affect on the market's behavior. In the continuum model, if the agents are identified with the points t on the line, an agent can disappear without changing the market at all; in the "infinitesimal" model, an agent can disappear without significantly changing the market. Which model is more appropriate depends in part on whether it is more sensible to draw the distinction between the significant and the insignificant or between something and nothing. We already have the invisible hand of the market; perhaps it is better to imagine very small traders rather than making them ghost-like as well.

The notion of infinitesimals is not entirely absent from standard mathematics. Algebraists prove, for instance, that there are ordered fields which are non-Archimedean, that is fields with an element w satisfying $w > 0$, $w > 1$, $w > 2$, etc. for all natural numbers. And since every element of a field must have an inverse, $1/w$ is in the field and $1/w$ must be smaller than $1/2$ and $1/3$ etc. for all the "standard" integers. Hence $1/w$ is an infinitesimal. One could give an algebriac proof showing that the real numbers can be embedded in a non-Archimedean field, thus making rigorous the

argument of Leibnitz and the intuitive notion of an "infinitesimal" trader. Yet this would not be a great enough advance to allow for an economic model. For even though we have extended the real numbers to include infinitesimals, we have not extended the functions on the real numbers to allow for infinitesimals. What is $\sin(x + dx)$, for instance? What is the utility, $U(x + dx)$? It is to allow for the definition of utilities on infinitesimals as well as standard real numbers that we rely on the model of "nonstandard analysis" created by Abraham Robinson in the early 1960's.

Mathematics mainly consists of choosing a class of objects to study, for instance the integers or the real numbers or more generally fields or Banach spaces and then discovering as many true statements as possible that apply to the designated class. For example it is true for every field that the relation of addition is commutative, which might be expressed $(\forall x)\ (\forall y)\ (x + y = y + x)$. Similarly we might express the principle true of the integers that every bounded set has a greatest element in the following way: $(\forall Q)((\exists x)(y \varepsilon Q \rightarrow y \leq x) \rightarrow (\exists z)(z \varepsilon Q \& (y \varepsilon Q \rightarrow y \leq z)))$. Note that this sentence is not true of the real numbers however. The first sentence, on the other hand, is true of both the integers and the real numbers.

It is customary when considering a language and its symbols to think only of the "reality" they describe. In the early 1900's it was observed, as we have just seen, that some statements in a language have precisely the same form when applied to different structures; that is, if the symbols are given different interpretations (($\forall x$) could be taken to mean "for all real numbers" instead of "for all integers") they still remain true. If we think of a language L as a collection of symbols ($\forall$ x,+,27,etc.) and rules of syntax for combining those symbols into statements, then we can think of an

interpretation of the language as a structure (made up of individuals, sets of individuals, sets of sets, etc.) such that to every symbol there corresponds a definite element of the structure (for instance the symbol + could be associated with the set of triplets of real numbers $[(x,y,z) \mid x + y = z]$, the symbol $(\forall x)$ could be taken as the set of all real numbers) such that with the given interpretation every statement is either true or false. Then given a language and a formal statement in the language, we call a model for that statement any interpretation for the language in which that statement is true. Many statements will be true under different interpretations, that is have many models, some statements are true under every interpretation (these are tautologies) and lastly the contradictory statements admit no models. Given a collection of statements, a model for that collection is an interpretation of the language in which all the statements are true. Thus in the language where symbols usually describe the real numbers, the collection of all theorems concerning the real numbers is modeled by the real number interpretation for that language. The real numbers are called the standard model.

In 1930 Kurt Gödel proved one of the most fundamental and surprising theorems of modern logic, the compactness theorem. His theorem proves that if $S = [S_t]$ is any finite or infinite collection of statements in a language L such that for any finite subset $A \subset S$ of statements we can find an interpretation for L that makes all the statements in A true, (that is a model for A), then we can find an interpretation for L that simultaneously makes all of the perhaps infinitely many statements of S true.

As a consequence of this theorem, Abraham Robinson was able to demonstrate

the existence of infinitesimals and nonstandard analysis. Let L be a language that adequately describes the real numbers and the properties they possess (so there is a symbol for each real number, eg 27, each function, eg sin, each relation, eg <, each subset, eg N, etc.). Let us suppose $S = [S_t]$ is the collection of all true statements (either discovered or so far undiscovered) about the reals that can be expressed in the language L. Of course the real number structure, which we designate $\mathfrak{R}$, is a model for S. Now let us expand the language L to include the new symbol w, and call it $\bar{L}$, and let us add to S the statements $w \in R$, $0 < w$, $1 < w$, $2 < w$, $3 < w$,... and in general $r < w$ for every real number r. We now have a new language $\bar{L}$ and an infinite (in fact uncountable) collection of statements $\bar{S} = S \cup \bigcup_{r \in R} [r<w] \cup [w \in R]$

It is easy to see, however, that for every finite subset A of $\bar{S}$ we can find an interpretation for $\bar{L}$ in which all the statements of A are true. For if A is finite then in particular there are only a finite number of sentences of the form $r < w$, hence we can take as our interpretation for $\bar{L}$ the standard real numbers structure $\mathfrak{R}$, letting w symbolize some natural number n_0 bigger than any of the r in A. Of course this is a model for A, but not for $\bar{S}$. Yet by the compactness theorem of Godel, since we can find a (different) model for each finite set of sentences A in $\bar{S}$, there must be some interpretation $^*\mathfrak{R}$ for $\bar{L}$ in which all the statements of $\bar{S}$ are true. And $\bar{S}$ contains S, hence $^*\mathfrak{R}$, called the nonstandard structure, must satisfy all the same properties as the standard structure. In particular, whatever is formally true of the interpretation of R in $\mathfrak{R}$ must also be true of the interpretation of R in $^*\mathfrak{R}$. Thus the statement $(\forall x \in R)(\forall y \in R)$ $(x + y = y + x)$ is still true when $(\forall x \in R)$ is interpreted to mean for all $x \in {}^*R$, where $^*R$ is the interpretation of the symbol R in the language $\bar{L}$ by the structure $^*\mathfrak{R}$. Likewise the statement $27 \in R$ is true in $^*\mathfrak{R}$, hence $^*R$ includes all the reals and we can therefore think of $^*R$ as an <u>extension</u> of the real numbers.

On the other hand, we know that $^*R$ cannot be the same as R for it must include an interpretation for the symbol w ; clearly whatever object is associated with w cannot also be associated with any other constant symbol r , since one of the statements in $\bar{S}$ is $r < w$ and one of the sentences included in S is $(\forall x \varepsilon R)(\forall y \varepsilon R)(x < y \rightarrow x \neq y)$. Thus w is infinitely large, since it is larger than any real number, and furthermore since the sentence $(\forall x \varepsilon R)(\exists y \varepsilon R)(x \cdot y = 1)$ is true of R , hence in S , it is true in $^*\mathfrak{R}$, hence there is a number $1/w \varepsilon {}^*R$ which is smaller than any standard real number, yet nonzero since $w \cdot 1/w = 1$. We can define an infinitesimal as any $x \varepsilon {}^*R$ such that $|x| < y$ for all standard positive real numbers y . We have created an infinitesimal; in fact an infinite number of them since if w is infinite, $w + 1$ must be infinite and so $1/(w+1)$ is another infinitesimal, etc.

Furthermore, we have also solved the problem mentioned earlier of extending functions from the standard reals to the infinite and infinitesimal reals. For imagine that $\hat{U}$ is a real-valued function defined on R . Then there is some symbol in L to describe $\hat{U}$, which we shall designate U , and it is certainly true that $(\forall x \varepsilon R)(\exists y \varepsilon R)(U(x) = y)$. Hence this statement is in S , and is therefore true in the nonstandard interpretation of $\bar{L}$, so for all x in $^*R$, $U(x)$ is in $^*R$. The nonstandard interpretation of U must therefore be a function which agrees with $\hat{U}$ on all the standard reals (since for example $U(27) = 35$ might be one of the statements in S) and is in addition defined on all the nonstandard reals as well. We would like to know whether this extension is arbitrary or if it satisfies some convenient property (other than those which could be described in the language L) such as if x is infinitesimally close to the standard y , then $U(x)$ is infinitesimally close to $U(y)$.

Before investigating this question, however, we must be careful to explain what we have accomplished. Any statement S_t which can be formulated in the language L can also be formulated in the language $\bar{L}$ and if it is true under the interpretation $\mathfrak{A}$ of L then $S_t \varepsilon S \subset \bar{S}$ and the formal statement S_t is by construction true in the interpretation $*\mathfrak{A}$ of $\bar{L}$, when the symbols of S_t are of course also interpreted by $*\mathfrak{A}$. (For example the symbol U in the language L is interpreted by $\mathfrak{A}$ to be a function defined on the standard reals whereas the same symbol U considered as part of the bigger language $\bar{L}$ is interpreted by $*\mathfrak{A}$ to be a function on all the nonstandard reals *R, including the ordinary reals.) We call this transferance of truth from S_t in $\mathfrak{A}$ to S_t in $*\mathfrak{A}$ the transfer principle.

The crucial point is that many of the symbols in L get unusual interpretations in $*\mathfrak{A}$ when considered as part of the language $\bar{L}$. For instance, the symbol N which in L was interpreted by $\mathfrak{A}$ to mean the standard natural numbers is interpreted by $*\mathfrak{A}$ to be a much bigger set. For the sentence noted earlier: $(\forall Q \subset N)((\exists x \varepsilon R)(y \varepsilon Q \rightarrow y \leq x) \rightarrow (\exists z \varepsilon Q)(y \varepsilon Q \rightarrow y \leq z)))$ expressing the idea that for any subset Q of N which is bounded above by some $x \varepsilon R$, there must be a largest element z in Q, is true in the interpretation $\mathfrak{A}$ of L and hence must be true in the interpretation $*\mathfrak{A}$ of $\bar{L}$. But we know that w is bigger than any standard natural number, hence N could not be interpreted by $*\mathfrak{A}$ to be simply the set of standard natural numbers, for then $Q = N$ would have a largest element and we know that the opposite is true since $(\forall x \varepsilon N)(x+1 \varepsilon N \& x+1>x)$ is in S. Thus N must be interpreted by $*\mathfrak{A}$ to include infinite integers as well as finite integers (we know that every standard natural number is included because for instance the statement $27 \varepsilon N$ is in S.) To facilitate discussion we shall denote by *N, and similarly for any symbol U in L by *U, the interpretation of N in $\bar{L}$ by $*\mathfrak{A}$. When there is no chance of

ambiguity, we shall always take the interpretation of N to be the standard natural numbers. Thus we have just seen that $*N \sim N$ is nonempty. Moreover, the same argument shows that the standard natural numbers N are not a set at all in the structure $*\mathfrak{A}$; for if they were they would be a bounded subset of $*N$ and hence have a greatest element! This paradoxical conclusion can be made easier to understand if we observe that the structure $*\mathfrak{A}$ is an interpretation of $\bar{L}$; $*\mathfrak{A}$ therefore necessarily must have an element corresponding to each symbol in $\bar{L}$. But the symbol N in $\bar{L}$ is interpreted by $*\mathfrak{A}$ as the nonstandard integers $*N$. There is no symbol in $\bar{L}$ which needs to be interpreted by the standard integers, hence it should not be so surprising that the structure $*\mathfrak{A}$ does not include the set of natural numbers.

We call all the subsets of $*R$ which are in the structure $*\mathfrak{A}$ the internal subsets of $*R$ and more generally any element of $*\mathfrak{A}$ an internal element of $*\mathfrak{A}$ (in measure theory only some of the subsets of R , the measurable sets, occur as elements in the structure). The subsets of $*R$ which are not internal, and hence so far as the structure $*\mathfrak{A}$ is concerned do not exist, are called external. Thus the set of standard natural integers N is an external subset of $*R$. Moreoever, in any statement such as $(\forall Q \subset N)((\exists x \in N)(y \in Q \rightarrow y \leq x) \rightarrow (\exists z \in Q)(y \in Q \rightarrow y \leq z)))$ the $(\forall Q \subset N)$ must be interpreted to mean for all internal subsets of N . The model $*\mathfrak{A}$ gives an interpretation to every symbol in $\bar{L}$ such that every statement in $\bar{S}$ is true; it does not necessarily give interpretations to sets which are not symbolized in the languages L or $\bar{L}$. On the other hand, there may be sets in $*\mathfrak{A}$ which do not correspond to symbols in $\bar{L}$, for instance $[x \in *N | x \leq w]$ is clearly internal, for it is described in the language $\bar{L}$ by $[x \in N | x \leq w]$, so we can imagine a single symbol for it. Let $*L$ be the extension of $\bar{L}$ which adds to L symbols for all the internal elements of $*\mathfrak{A}$. Then $*\mathfrak{A}$ is an interpretation for $*L$.

To see the significance of internality, suppose that we imagine some $Q \subset {}^*N$ and suppose that f is a function $f: Q \to {}^*R$ and finally suppose that there is some $w \in {}^*R$ such that $n \leq w$ for all $n \in Q$. Then if Q is an internal element of ${}^*\mathcal{O}$, that is it can be described in the language ${}^*L$, and if f is an internal function, so that it too can be described in ${}^*L$, then f reaches a maximum at some $n_0 \in Q$. On the other hand, if Q is not internal then there is no guarantee that f reaches a maximum on Q. We shall often have occasion, in what follows, to define a Q and an f we hope will reach a maximum on Q; the argument is always completed by proving that Q is internal.

We should also note that the set of infinitesimals $I = [x \in {}^*R \mid x \simeq 0]$ is also an external set, for if I were internal, then the set of standard integers $N = [x \in {}^*N \mid 1/x \notin I]$ would also be internal. In what follows we shall often prove theorems about sets which are defined by infinitesimals and hence are external, as well as proving theorems about internal sets. In fact the nonstandard core and the nonstandard Bargaining Set are both external sets. It is precisely for this reason that it is interesting to study them in a nonstandard economy. We shall explain this idea when we present the nonstandard model of an economy in the next section.

We are now in a position to return to our original question of investigating the arbitrariness of the extension ${}^*U$ of U to the nonstandard reals. We say that x and y are infinitesimally close $x \simeq y$ if $|x-y|$ is infinitesimal. It is trivial to see that $\simeq$ is an equivalence relation. But this implies that for any finite nonstandard x (that is any $x \in {}^*R$ such that there is a $M \in R$ with $x \leq M$) we can find a unique ${}^\circ x \in R$, called the standard part of x, such that ${}^\circ x$ is infinitesimally close to x. If y and z are both in R, then if $y \simeq x$ and $z \simeq x$, then $y \simeq z$ which is impossible unless $y = z$ since

y and z are both standard. Hence ${}^{\circ}x$, if it exists, is unique. But it must exist, for consider $A = [r \in R \mid r \leq x]$. This set must have a least upper bound in R , call it y . Then $y \simeq x$, for if not then $|x - y| > \delta \in R$. Then either $y < x$ and $y + \delta$ is still $< x$, contradicting the fact that y is an upper bound for A , or else $y > x$ and $y - \delta$ is still $> x$, contradicting the choice of y as the least upper bound of A .

Now, given a standard function $U\colon R \to R$ we say that U is continuous at a , or that U converges at a , if (for all $\varepsilon > 0)(\exists \delta > 0)(|a-x| < \delta \to |U(x)-U(a)| < \varepsilon)$. We claim that if U is continuous at some $a \in R$, then $|{}^{*}U(x)-U(a)|$ is infinitesimal for all x infinitesimally close to a . This is easy to verify, for to every standard $\varepsilon > 0$ there is a standard $\delta(\varepsilon)$ such that $|x-a| < \delta(\varepsilon) \to |U(x) - U(a)| < \varepsilon$. Since this statement is true in $\mathfrak{A}$, it must be true in ${}^{*}\mathfrak{A}$ that $|x-a| < \delta(\varepsilon) \to |U^{*}(x)-U(a)| < \varepsilon$. If $x \simeq a$, then for any standard $\delta(\varepsilon)$, $|x-a| < \delta(\varepsilon)$, so therefore $|{}^{*}U(x) - U(a)| < \varepsilon$ for all standard $\varepsilon > 0$. This is precisely that $U(x) \simeq U(a)$ whenever $x \simeq a$. Conversely, if whenever $x \simeq a$, $U(x) \simeq U(a)$, then it is certainly the case that for any fixed standard $\varepsilon > 0$ the statement $(\exists \delta > 0)(|x - a| < \delta \to |U(x) - U(a)| < \varepsilon)$ is true in ${}^{*}\mathfrak{A}$, since we could take δ infinitesimal and then by hypothesis $|x-a| < \delta \to x \simeq a \to U(x) \simeq U(a) \to |U(x) - U(a)| < \varepsilon$. But the above statement is expressible in the language L , hence it is true in the interpretation $\mathfrak{A}$ if and only if it is true in the interpretation ${}^{*}\mathfrak{A}$ of $\bar{L}$. Thus we have shown that for all $\varepsilon > 0$, $\varepsilon \in R$ the statement $(\exists \delta > 0)(|x-a| < \delta \to |U(x)-U(a)| < \varepsilon)$ holds. We have shown that the two conditions $[x_n \to x$ implies $U(x_n) \to U(x)]$ and $[x \simeq a$ implies ${}^{*}U(x) \simeq U(a)]$ are equivalent.

Thus we have verified that for continuous functions $U\colon R \to R$, the

nonstandard extension $*U$ of U has the convenient property that if $x \simeq a$, then $*U(x) \simeq U(a)$. We can return to the original problem posed by Berkeley and note that if U is differentiable at a, then $\lim_{dx \to 0} \frac{U(a+dx) - u(a)}{dx} = U^1(a)$, hence by what was shown above, for any nonzero infinitesimal dx, $\frac{U(a+dx) - U(a)}{dx} \simeq U^1(a)$, that is if $U^1(a)$ exists, then we can form the difference $d\,U = U(a+dx) - U(a)$, then divide by the nonzero infinitesimal dx, getting $\frac{du}{dx}$ and then taking the standard part of $\frac{du}{dx}$ we get ${}^\circ\frac{du}{dx} = U^1(a)$. To put it differently, if $dx \simeq 0$, then $U(a+dx) = U(a) + U^1(a)dx + \alpha(dx)$ where $\frac{\alpha(dx)}{dx} \simeq 0$.

We give one more example of the methods of nonstandard analysis which will prove useful to us in later sections. Suppose that $[x_n]$ is an internal sequence of elements in $*R$ such that $x_n \simeq 0$ for all finite n. Then the prolongation theorem states that there is some $r \varepsilon *N \sim N$ such that $x_r \simeq 0$. To verify this, observe that if $x_n \simeq 0$ for n finite, then by definition $|x_n| < 1/n$ for all standard natural numbers n. Hence the internal set $[n \varepsilon *N \mid |x_n| < 1/n]$ includes all the standard integers; since the standard integers are not an internal set, it must also include some infinite integers r. But then $|x_r| < 1/r \simeq 0$ and the prolongation theorem is verified.

We are now ready to set down the nonstandard model, discovered in 1971 by Brown and Robinson, of a perfectly competitive economy. We remark first that everything that has been done so far could equally have been applied to R^ℓ, the standard ℓ dimensional vector space, instead of R. We denote the nonstandard interpretation of R^ℓ by $*R^\ell$, we let $*R^\ell_+ = [x \varepsilon *R^\ell \mid x_i \geq 0$ for all $i = 1 \ldots \ell]$. If x and y are vectors in $*R^\ell_+$ we write that $x \simeq y$ if

$x_i \simeq y_i$ for $i=1\ldots,\ell$. We mean by $x \geqq y$ that $x_i \geqq y_i$ for all i ; $x > y$ means $x \geqq y$ and for some i , $x_i > y_i$; $x >> y$ means $x_i > y_i$ for all i . By $x \gtrsim y$ we mean that $x_i \geqq y_i$ or $x_i \simeq y_i$ for all i ; $x \underset{\not\approx}{\geq} y$ means $x \gtrsim y$ and for some i , x_i is greater by a noninfinitesimal amount than y_i , that is the scaler $x_i \overset{>}{\not\approx} y_i$; $x \underset{\not\approx}{>>} y$ means $x_i \overset{>}{\not\approx} y_i$ for all i .

By the norm of a vector $x = (x_1,\ldots,x_\ell)$ we mean the sup norm $||x|| = \max_i |x_i|$. A nonstandard vector x is said to be finite or near standard if $||x|| \leq n$ for some standard number n . If x is finite, then as we showed earlier there is a unique standard vector, called the standard part of x and denoted $°x$ such that $°x \simeq x$.

Before developing our model, we summarize what we have done in this section. We showed that the notion of perfect competition, like the original idea of differentiation in calculus conceived by Leibnitz, presupposes the existence of agents which are negligibly small but nonvanishing. This apparent paradox is resolved by modern mathematical economics with measure theory and limit arguments; nonstandard analysis solves the same paradox by extending the set of real numbers R (and indeed the entire real structure $\mathfrak{A}$ of subsets of R , subset of subsets, functions on R , relations etc.) to include infinitesimals satisfying all the same properties as R . Instead of conceiving of a language L as designed to describe a model $\mathfrak{A}$, we begin with the language L and we find another structure or model $^*\mathfrak{A}$ that interprets the symbols of L so that precisely those statements S_t which are true in the interpretation $\mathfrak{A}$ are true when interpreted by $^*\mathfrak{A}$.

[1]For instance, Abraham Robinson Nonstandard Analysis.

The fact that by construction the properties of $\mathfrak{A}$ and $^*\mathfrak{A}$ are formally identical, when expressed in the language L , is called the transfer principle. Next we showed that $^*\mathfrak{A}$, which we call the nonstandard model of the reals, interprets the symbol R to include infinitesimal and infinite numbers as well as all the standard real numbers. Moreover, we showed that every standard real function $\hat{u}$ is extended in a natural way to the nonstandard reals in the interpretation $^*\mathfrak{A}$. Thus if $\hat{u}$ is a standard real valued function in $\mathfrak{A}$ it is described in the language L by a symbol u which is interpreted by the nonstandard model $^*\mathfrak{A}$ to be a function $^*u$ that agrees with $\hat{u}$ on all the standard reals. If $\hat{u}$ is continuous we showed that the extension to $^*u$ is particularly convenient: if dx is infinitesimal and x is a standard real number, then $^*u(x+dx)$ is infinitesimally close to $\hat{u}(x)$.

Last we showed that some sets of elements of $^*\mathfrak{A}$ are not themselves elements of $^*\mathfrak{A}$. These are called external sets. In fact the set of standard real number itself and the Bargaining Set and the Core we shall define are all external sets. Thus we are interested in proving theorems about subsets of the nonstandard structure $^*\mathfrak{A}$.

Nonstandard analysis was conceived originally as a powerful tool for helping mathematicians to understand standard analysis, that is to discover and prove in a simple fashion theorems about $\mathfrak{A}$. For that purpose three techniques are generally invoked; the transfer principle, internality (externality), and concurrence (which we do not use in this paper). For example with nonstandard analysis one can give a one line proof of Tychonoff's compactness theorem, and Robinson and Bernstein were able to solve an open question on the invariance of subspaces of Hilbert space.

More recently Don Brown and others have shown that it is possible to construct rigorous models of intuitive idealizations of economic relationships not only using $\mathfrak{A}$ but alternatively using subsets of the nonstandard structure $^*\mathfrak{A}$. We try to show in the next two sections that the nonstandard model is often more natural and its proofs are simpler than any of the standard models of the competitive hypothesis. In this light we see that nonstandard analysis may not only be a useful technique for discovering theorems about $\mathfrak{A}$, but also for providing a rigorous framework for modelling economic notions not completely captured in the standard models or even not formalizable in the standard structure.

For the sake of completeness, we quickly explain the idea of concurrence. We created the nonstandard model by using the relation $<$ and introducing a new symbol w and statements of the form $r < w$. Given any finite collection A of statements we could always find a model for A in $\overline{L}$ by interpreting w as a big enough standard real number. This was possible because the relation $<$ is concurrent: a relation R is concurrent if given a finite set of elements $X_1, \ldots X_n$ in the domain of R, we can always find a standard y such that $X_1 Ry, \ldots, X_n Ry$ (in case R is $<$, take $y = \max (y_1, \ldots, y_n) + 1$). There is no reason not to introduce a number symbol w_R for every concurrent relation R and all statements of the form XRW_R for all X in the domain of R, for all concurrent relations R. Then the language $\overline{L}$ is much bigger than the language L, but exactly the same logic as before shows that there must be a model for $S \cup [x \, R \, w_R]$ in $\overline{L}$, and hence a very big nonstandard model of S in L.

SECTION VI

The Model

Here we set down our definition of a nonstandard economy E and the associated nonstandard games $(T,V,E,[U_t]_{t\varepsilon T})$. We make rigorous the competitive hypothesis by positing an economy with an infinite number of agents, traders who have finite initial endowments and bounded utility functions, and preferences which do not depend on infinitesimal variations in consumption. We conclude by proving that any nonstandard economic game $(T,V,E,[U_t]_{t\varepsilon T})$ with bounded differentiable utilities (an assumption made by both Aumann and Brown-Loeb in their value equivalence papers) can be approximated by a type economic game $(T,\bar{V},\bar{E},[\bar{U}_t]_{t\varepsilon T})$ in which every trader is one of V types, where $V/(\#$ traders in $E)$ is infinitesimal, in such a way that the Core and Bargaining Set of (T,V) are the same as the Core and Bargaining Set of $(T,\bar{V})$, respectively.

We make precise the notion of a large number of traders by letting T, the set of traders, have an infinite number of elements, that is let $T = [n\varepsilon {}^*N \;\; n \leq w]$ where w is an infinite integer, $w\varepsilon {}^*N \sim N$.[1] Note that we have proved ${}^*N \sim N$ is nonempty.

A coalition is an internal subset of T.

A coalition S is negligible if $|S|/w \simeq 0$.

All agents are assumed to have the same consumption set ${}^*R_+^{\ell}$.

An assignment is an internal map from T into ${}^*R_+^{\ell}$.

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[1] This model is due to Brown and Robinson [1971]. We use the same notation as Brown-Loeb 1976.

A trader is defined by his initial endowment an element of $*R_+^\ell$, and his preference relation, a binary relation on $*R_+^\ell$, so that a nonstandard exchange economy E is a set of triples $(t, I(t), >_t)_{t\varepsilon T}$ that assigns endowments and preferences in an internal way.

An assignment $[x(t)]_{t\varepsilon S}$ is strictly feasible for a coalition S if $\sum_{t\varepsilon S} x(t) \leqq \sum_{t\varepsilon S} I(t)$, and feasible if $\frac{1}{|S|} \sum_{t\varepsilon S} x(t) \lesssim \frac{1}{|S|} \sum_{t\varepsilon S} I(t)$. Observe that if S is nonnegligible, then $[x(t)]_{t\varepsilon S}$ is feasible iff $\frac{1}{w} \sum_{t\varepsilon S} x(t) \lesssim \frac{1}{w} \sum_{t\varepsilon S} I(t)$.

If $>_t$ is a preference relation over $*R_+^\ell$, then a utility function for $>_t$ is an internal map U_t: $*R_+^\ell \to *R^\ell$ such that for all $x,y\varepsilon *R_+^\ell$, $x >_t y$ iff $U_t(x) > U_t(y)$. We shall always assume that $U_t \varepsilon * \mathcal{C}'$, the nonstandard extension of the set of continuously differentiable functions, and therefore that the preferences themselves can always be represented by differentiable utilities. An internal family of utilities $[U_t]_{t\varepsilon T}$ is a representing family for $[>_t]_{t\varepsilon T}$ if for each $t\varepsilon T$, U_t is a utility function for $>_t$.

We define the core of a nonstandard economy $E = (t, I(t), >_t)_{t\varepsilon T}$ as any feasible assignment $[x(t)]_{t\varepsilon T}$ for T such that there is no blocking coalition or objection $(S,[y(t)]_{t\varepsilon S})$ satisfying:

(1) S is nonnegligible and internal

(2) $\frac{1}{|S|}\sum_{t\varepsilon S} y(t) \lesssim \frac{1}{|S|}\sum_{t\varepsilon S} I(t)$, that is $[y(t)]_{t\varepsilon S}$ is feasible for S

(3) $y(t) \gg_t x(t)$, that is for all $y \simeq y(t)$ and all $x \simeq x(t), y(t) >_t x(t)$.

We can associate with an economy $E = (t, I(t), >_t)_{t\varepsilon T}$ and any representing family of utility functions $[U_t]_{t\varepsilon T}$ a transferable utility game $(T,V) = (T,V,E,[U_t]_{t\varepsilon T})$ where V is an internal $^*R$ valued function defined on the coalitions of T by: $V(S) \equiv \text{Max} [\sum_{t\varepsilon S} U_t(x(t))$ s.t. $\sum_{t\varepsilon S} x(t) \leqq \sum_{t\varepsilon S} I(t)$ and $x(t) \varepsilon {}^*R^{\ell}_{+}$ for all $t \varepsilon T]$. Observe that different representing families $[U_t]_{t\varepsilon T}$ give rise to different games associated with the same economy.

Proposition: The function V in $(T,V,E,[U_t]_{t\varepsilon T})$ is well-defined and internal (that is the maximum is achieved for all S).[1]

Proof: The proof follows by transfer from the finite case. QED This fundamental theorem is a triviality to prove in the nonstandard world whereas in other models such as the atomless measure theoretic model of Aumann it requires a lengthy and difficult proof. This result alone is a significant justification of the use of nonstandard economies to model perfectly competitive systems and it illustrates a more basic principle. We study idealized perfectly competitive economies, like the nonstandard and continuum models, in order to obtain approximate information about large finite economies. Therefore we

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*Note also that for S and R disjoint, $V(S \cup R) \geqq V(S) + V(R)$.

must cast certain concepts (such as the associated transferable utility games or the value) which are natural for finite games into a form we hope is appropriate to our infinite model; unfortunately for the continuum model this is often difficult to do but for the nonstandard model it can always be accomplished trivially by the transfer principle.

We define a <u>strict payoff configuration</u> for any game (T,V) as an internal function $x: T \to {}^*R_+$, which we denote by $[X_t]_{t\in T}$, satisfying $\sum_{t\in T} x_t \leq V(T)$. If $\frac{1}{|T|}\sum_{t\in T} x_t \lesssim \frac{1}{|T|} V(T)$ we call $[x_t]_{t\in T}$ a <u>payoff configuration</u>. Similarly we define a <u>strict imputation</u> $[x_t]_{t\in T}$ for (V,T) as a strict payoff configuration satisfying $\sum_{t\in T} x_t = V(T)$ and an <u>imputation</u> as a payoff configuration $[x_t]_{t\in T}$ satisfying $\frac{1}{|T|}\sum_{t\in T} x_t \simeq \frac{1}{|T|} V(T)$.

<u>Proposition 2</u>: There is a unique function Φ which associates with any game, standard or nonstandard (T,V), a strict payoff configuration $\Phi(T,V) = [\phi_t]_{t\in T}$ such that Φ agrees with the value payoff configuration for all finite games. If (T,V) is nonstandard we call $\Phi(T,V) = [\phi_t]_{t\in T}$ the <u>strict value payoff configuration</u> of (T,V) and we can express ϕ_t by the formula

$$\phi_t = \sum_{n=0}^{w-1} \frac{1}{w} \sum_{\substack{S \not\ni t \\ |S|=n}} \frac{1}{\binom{w-1}{n}} (V(S[t]) - V(S)) .$$

Moreoever, the strict value is a strict imputation.

<u>Proof</u>: Let $H \in$ [illegible] be the set of all standard games, $H = [T,V]$.

Let G be the set of all payoff configurations $[(T,x)]$. We know that for finite games there is a unique value function $\phi;\ H \to G$ such that $\phi(T,V) = (T,x)$, $\sum_{t\in T} x_t = V(T)$, and x_t is given by the above formula. But now consider the nonstandard structure *[illegible] and the elements (in *[illegible]), ${}^*H$, ${}^*G$, and ${}^*\phi$

corresponding to H, G, and Φ respectively. By transfer $*\Phi$ is defined on all of $*H$. Thus by transfer the value payoff configuration for any nonstandard game exists and the value payoff configuration for any game (T,V) is a strict imputation. Alternatively, we could have proved directly that the formula given in the theorem is internal. QED

We say that $[x_t]_{t\varepsilon T}$ is a <u>value payoff configuration</u> for a nonstandard game (T,V) if $x_t \simeq \phi_t$ for all $t\varepsilon T$ where $[\phi_t]_{t\varepsilon T}$ is the strict value payoff configuration. We say that $[x_t]_{t\varepsilon T}$ is a <u>competitive payoff configuration</u> for $(T,V,E,[U_t]_{t\varepsilon T})$ if there is a feasible assignment $[x(t)]_{t\varepsilon T}$ and a price vector $p \underset{\sim}{>>} 0$ such that $x_t \simeq U_t(x(t)) - p^t(x(t) - I(t)$ and for all $y \varepsilon *R^\ell_+$, $U_t(x(t)) - p^t(x(t) - I(t)) \gtrsim U_t(y) - p^t(y - I(t))$. If in addition $x_t \simeq U_t(x(t)) - p^t(x(t) - I(t)) \simeq U_t(x(t))$ for all $t\varepsilon T$, that is for all $t\varepsilon T$, $p^t x(t) \simeq p^t I(t)$, then we call $[x_t]_{t\varepsilon T}$ a <u>natural competitive payoff configuration</u>. We can define a <u>value allocation</u> for the nonstandard economy $E = (t, I(t) >_t)_{t\varepsilon T}$ as any feasible assignment $[x(t)]_{t\varepsilon T}$ for T for which there is an associated game $(T,V,E,[U_t]_{t\varepsilon T})$ in which $[x_t]_{t\varepsilon T} \simeq [U_t(x(t))]_{t\varepsilon T}$ is a value payoff configuration. We define a <u>competitive equilibrium</u> for the nonstandard economy E as any feasible assignment $[x(t)]_{t\varepsilon T}$ and a price vector $p \underset{\sim}{>>} 0$ such that for all $t\varepsilon T$, $p^t x(t) \lesssim p^t I(t)$ and if for some $y \varepsilon *R^\ell_+$, $y \underset{\sim}{>>}_t x(t)$, that is if for all $\bar{y} \simeq y$ and $\bar{x} \simeq x(t)$, $\bar{y} >_t \bar{x}$, then $p^t y \underset{\sim}{>} p^t I(t)$.[1] Of course if $[x(t)]_{t\varepsilon T}$ gives rise to a <u>natural</u> competitive

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[1] It is common in the literature (Brown-Robinson, Brown-Loeb) to define the competitive equilibria and the value allocations "up to negligible coalitions" as well, that is a feasible assignment $[x(t)]_{t\varepsilon T}$ is a competitive equilibria if there is some negligible set $\bar{R}$ such that for all $t\varepsilon T \sim \bar{R}$, $x(t)$ is maximal (as we have defined it). A value allocation is defined as a feasible assignment $[x(t)]_{t\varepsilon T}$ such that there is some associated game $(T,V,E,[U_t])_{t\varepsilon T}$ in which $(U_t(x(t)) \simeq \phi_t(V)$ for all $t\varepsilon T \sim \bar{R}$. This kind of definition is necessary to prove the existence of a competitive equilibrium and the equivalence between core allocations in E and competitive allocations in E, but since we shall not be concerned with those problems it is conceptually easier to use our definitions.

payoff configuration for some associated game $(T,V,E,[U_t]_{t\in T})$, then $[x(t)]_{t\in T}$ is a competitive equilibrium for T. To prove the converse, that given any competitive equilibrium $[x(t)]_{t\in T}$ for the nonstandard economy E , we can find an associated game $(T,V,E,[U_t]_{t\in T})$ in which $[U_t(x(t))]_{t\in T}$ is a natural competitive payoff configuration requires stronger assumptions on E , for instance that the $>_t$ are convex and can be represented by concave utility functions. A proof similar to the finite case could be given. In this paper we shall not need to require concave utilities in order to prove our Bargaining Set Equivalence.

We define the core of a transferable utility economic game $(T,V,E,[U_t]_{t\in T})$ as the set of feasible payoff configurations $[x_t]_{t\in T}$ such that there is no blocking coalition or objection $(S,[y_t]_{t\in S})$ satisfying

(1) S is nonnegligible and internal

(2) $\frac{1}{|S|}\sum_{t\in S} y_t \lesssim \frac{1}{|S|} V(S)$, that is $[y_t]_{t\in S}$ is feasible for S

(3) $y_t \gtrsim\!\!\!\!/\; x_t$ for all $t\in S$

Note that S can be used as a blocking coalition for $[x_t]_{t\in T}$ if and only if S is nonnegligible and $\frac{1}{|S|} V(S) \gtrsim\!\!\!\!/\; \frac{1}{|S|}\sum_{t\in S} x_t$, for we can take $y_t = x_t + \frac{1}{|S|}[V(S) - \sum_{t\in S} x_t]$ for all $t\in S$. But this in turn implies, since S is nonnegligible, that $\frac{1}{w} V(S) \gtrsim\!\!\!\!/\; \frac{1}{w}\sum_{t\in S} x_t$. For any $S\subset T$, define $x(S) = \sum_{t\in S} x_t$. Then S blocks $[x_t]_{t\in T}$ means S is nonnegligible and $\frac{1}{w} V(S) \gtrsim\!\!\!\!/\; \frac{1}{w} x(S)$. Actually we shall show later in this section that if S is negligible, $\frac{1}{w} V(S) \simeq 0$. Hence S blocks $[x_t]_{t\in T}$ if and only if $\frac{1}{w} V(S) \gtrsim\!\!\!\!/\; \frac{1}{w} x(S)$ and a payoff configuration $[x_t]_{t\in T}$ is in the core of $(T,V,E,[U_t]_{t\in T})$ if and only if $\max_{S\subset T} \frac{1}{w}[V(S) - x(S)] \simeq 0$. Observe that by

transfer from the finite case, the maximum must exist and further that since $\frac{1}{w}(V(T) - x(T)) \gtrsim 0$, the maximum can never be a standard amount less than zero.

We define the <u>δ-Bargaining Set</u>, keeping to the spirit of the original Aumann-Davis definition but making it appropriate to large economies and the competitive hypothesis as explained in sections 2 and 3, as the set of those payoff configurations $[x_t]_{t\in T}$ for (T,V) such that to any <u>objection</u> $(S,[y_t]_{t\in S},K)$ satisfying

(1) $K\subset S\subset T$, K,S internal, S nonnegligible

(2) $\frac{|K|}{|T|} \leqq \delta$

(3) $\frac{1}{|S|} \Sigma y_t \leq \frac{1}{|S|} V(S)$ (4) $y_t \gtrsim x_t$ for all $t\in S$

there exists a <u>counterobjection</u> $(R,[w_t]_{t\in R})$ satisfying

(1) $R\subset T$, $R\cap K = \emptyset$

(2) $\frac{1}{|R|} \sum_{t\in R} w_t \leq \frac{1}{|R|} V(R)$

(3) $w_t \geqq y_t$ for all $t\in R\cap S$ and $w_t \geqq x_t$ for all $t\in R\sim S$.

The Bargaining Set of any game (T,V) is defined as the union of all the δ-Bargaining Sets for all noninfinitesimal δ, $\beta S(T,V) = \bigcup_{\delta \gtrsim 0} \beta S_\delta$.

Observe that the core is always contained in the Bargaining Set. Note also that if $\delta_1 < \delta_2$, then $\beta S_{\delta_1} \supset \beta S_{\delta_2}$ since the smaller δ is, the easier to counterobject. For $\delta = 1$, it is easy to see that the δ-Bargaining Set is identical to the core for let $[x_t]_{t\in T}$ be a payoff configuration not contained in the core of $(T,V,E,[U_t]_{t\in T})$. Then $\max_{S\subset T} \frac{1}{w}[V(S) - x(S)] = D \gtrsim 0$. Let $0 \lesssim \varepsilon \lesssim D$ and find S_0 that solves $\max_{S\subset T} \frac{1}{w}[V(S) - x(S) + \varepsilon|S|]$. Then since for any R with $R\cap S_0 = \emptyset$,

$V(R \cup S_0) \geq V(R) + V(S_0)$, it follows that for such R, $V(R) - x(R) + \epsilon|R| \leq 0$, hence that $\frac{1}{|R|}V(R) \leq \frac{1}{|R|}x(R) - \epsilon$ whenever $R \cap S_0 = \emptyset$. Take $K = S_0$ and $y_t = x_t + \frac{1}{|S_0|}[V(S_0) - x(S_0)]$. Observe that since $\epsilon \lneq D$, $\frac{1}{|S_0|}[V(S_0] - x(S_0)] \gneq 0$, so $(S_0, [y_t]_{t\epsilon S}, K)$ is an objection. Moreover, if $(R, [w_t]_{t\epsilon R})$ is a counterobjection, $R \cap K = \emptyset$ means $R \cap S_0 = \emptyset$, hence by construction $\frac{1}{|R|}V(R) \leq \frac{1}{|R|}x(R) - \epsilon$, hence R is not a counterobjection after all and $[x(t)]_{t\epsilon T}$ is not in βS_1.

On the other hand, if δ is infinitesimal then the δ-Bargaining Set includes all the payoff configurations. For let $[x_t]_{t\epsilon T}$ be a payoff configuration and suppose $(S, [y_t]_{t\epsilon S}, K)$ is an objection, so that S is nonnegligible and $\frac{|K|}{|S|} \approx 0$. Then we shall show in section 7 that not only is $\frac{1}{|S|}V(K) \approx 0$, but $\frac{1}{|S|}[V(S) - V(S\sim K)] \approx 0$, hence putting $R = S\sim K$, $\frac{1}{|R|}y(R) \lesssim \frac{1}{|S|}V(S) \lesssim \frac{1}{|S|}V(S) \approx \frac{1}{|R|}V(S\sim K) \equiv \frac{1}{|R|}V(R)$ and $(R, [y_t]_{t\epsilon R})$ is a counterobjection. It is the aim of this paper to prove:

<u>Theorem</u>: For any $\delta \gneq 0$, the δ-Bargaining Set and the Core of any economic game $(T, V, E, [U_t]_{t\epsilon T})$ satisfying assumptions 1-3 are identical.

Observe that the purpose in defining all the solution concepts in terms of feasibility instead of strict feasibility is essentially to identify payoff configurations that differ only by an infinitesimal. That is, if $[x_t]_{t\epsilon T}$ and $[y_t]_{t\epsilon T}$ are payoff configurations satisfying $x_t \approx y_t$ for all $t\epsilon T$, then according to our definitions either both are core payoff configurations or neither is, either both are in the Bargaining Set or neither is and so on. If we had required strict feasibility then $[x_t]_{t\epsilon T}$ could be in the Bargaining Set while $[y_t]_{t\epsilon T}$ is not. However, changing all the definitions from feasibility to strict feasibility would not affect which equivalence classes of payoff configurations were in the core or Bargaining Set. That is, $[x_t]_{t\epsilon T}$

is in the core or Bargaining Set defined with feasibility if and only if there is some $[y_t]_{t\in T}$, $y_t \tilde{\sim} x_t$ for all t , which is in the core or Bargaining Set, respectively, defined with strict feasibility. The fundamental idea, which rigourizes the perfect competition hypothesis, is that preference is defined by noninfinitesimal differences $y_t \succ x_t$ or $y(t) \gg_t x(t)$. Once we show that the disappearance of a single trader t_0 from a coalition S can only make a finite difference $V(S) - V(S^{\sim}[t_0])$ (if $(T,V,E,[U_t]_{t\in T})$ satisfies assumptions 1-3 to be stated) then spreading out the finite difference over an infinite number of traders can only change each individual's payoff infinitesimally which by the way we have defined preferences does not disturb any of the traders enough to cause him to change his behavior. The point of using feasibility instead of strict feasibility is aesthetic - it does not affect the equivalence results whereas the way preferences are defined is fundamental to the competitive hypothesis embodied in large economies.

At this point it is crucial to observe that the strict imputations and strict value payoff configurations are examples of internal notions. They can be expressed in the standard language L and the properties, for instance the existence of the value, which hold in the standard finite games must therefore hold by transfer in the nonstandard games. But other notions, for instance the nonstandard core or feasibility rather than strict feasibility cannot be expressed in the language L since they depend on membership in the set of infinitesimals which we pointed out in the last section is not an internal set. Thus the nonstandard core cannot be described in the language L ; it is not even in the structure $*\mathfrak{M}$. But this is precisely what makes it useful to study. If we simply transferred the standard definitions of the core and Bargaining Set of a game $(T,V,E,[U_t]_{t\in T})$

the equivalence would not hold, for the statement $(\forall n \in N)\ (\exists (T,V,E,[U_t]_{t\in T}))$ $(|T| = n\ \&\ \beta S(T,V,E,[U_t]_{t\in T}) \neq C(T,V,E,[U_t]_{t\in T}))$ meaning that no matter how big we take n , we can find a standard game based on an economy with n traders in which the Bargaining Set and the Core differ, is true in $\mathcal{O}\!\!\mathcal{J}$ (although the Bargaining Set and Core may be "converging" as $n \to \infty$) and hence holds by transfer for nonstandard economies. In order to obtain an equivalence result we must allow for infinitesimal changes and hence an external core and Bargaining Set. On the other hand this allows us, as we show in section 9, to derive convergence results for sequences of finite economies (recall that $x_n \to x$ if and only if $x_n \tilde{\sim} x$ for all $n \in {}^*N \sim N$) . Moreover, as we shall show shortly, if one trader can only induce an infinitesimal change in the market or more generally an infinitesimal change in the environment facing any other trader, and if all traders are indifferent about infinitesimal perturbations then we can claim to have rigorously modeled the competitive hypothesis that one trader by his actions alone cannot significantly affect another.

In order to assure ourselves of a rigourous interpretation of the perfectly competitive hypothesis we must build into our model what we called in the introduction the essential assumption that agents are more alike than they are different. This assumption, which is common to all the large equivalence proofs in the literature, will amount basically to assuming that the characteristics of agents (initial endowments and preferences), though they may differ widely, stay bounded while the number of agents approaches infinity.[1] Intuitively, the consequence is that when the economies are small there appears to be no connection with the replicated model of Scarf-Debreu and Shapley, but when the number of agents increases without bound definite trader types emerge such that the ratio of the number of

[1]Without this assumption, some agent may grow as fast as the economy and always exert a nonnegligible influence on the other traders.

types (which itself is becoming infinite) to the number of agents in the economy nevertheless goes to zero. In the end we get almost a type economy with unequal replications (more traders of one type than another), and it should not be hard to believe that unequal replications do not affect the equivalences which held with equal replications. Indeed, the proof we originally envisaged for the Bargaining Set equivalence confirms rigorously this intuition and can be applied to proving the value equivalence and core equivalence as well. In this paper we give a simpler and more direct proof (and as an aside we show how to construct a type economy approximation to any given economy).

We shall restrict our attention to economies $E = (t, I(t), >_t)_{t\varepsilon T}$ and representing families of utilities $[U_t]_{t\varepsilon T} \subset {}^*\mathcal{U}$ satisfying

<u>The Assumptions</u>

(1) $|T| = w\varepsilon {}^*N \sim N$ so there will be an infinite number of traders

(2) $I(t)$ is finite and $I(t) \underset{\not\approx}{\gg} 0$ for all $t\varepsilon T$. Hence there is a maximum finite amount of any good that any trader gets as his initial endowment, and also each trader starts with a non-infinitesimal amount of every good. Thus the initial endowments can be very different, but their variation is bounded by some finite number.

(3) Each $>_t$ can be represented by a $U_t \varepsilon {}^*\mathcal{U}$ where $\mathcal{U}$ is a (perhaps infinite) set of standard utilities $[U]$ satisfying the following three boundedness properties:

(a) The U in $\mathcal{U}$ are C^1 functions from R^ℓ_+ to R,

(b) The U's are uniformly bounded, that is there is some $M\varepsilon R^\ell$ such that $M \geq U(x)$ for all $U \varepsilon \mathcal{U}$, $x \varepsilon R^\ell_+$.

(c) The gradients ∇U are, on compact sets, uniformly bounded and uniformly positive, i.e. for every compact set $F \subset R^{\ell}_{+}$ there exist vectors $\bar{a}$ and $\bar{b} \in R^{\ell}$ such that $0 \ll \bar{a} \leq \nabla U(x) \leq \bar{b}$ for all $U \in \mathcal{U}$, $x \in F$.

Note that by transfer all these statements hold for U in ${}^*\mathcal{U}$. These are the same assumptions made by Aumann and Brown-Loeb in their value equivalence papers.[1] If in addition the representing family of utilities $[U_t]_{t \in T}$ satisfies

(d) $U_t(y) \leq U_t(x) + \nabla U_t(x)(y-x)$ for all $x, y \in {}^*R^{\ell}_{+}$ and for all $t \in T$, then we write that we have a concave family of representing utilities.[2]

The main consequence of these uniform boundedness assumptions is summarized in the following theorem:

Proposition 1 (Brown-Loeb): Let $((t, I(t), >_t))_{t \in T}$ be a nonstandard economy satisfying assumptions 1-3 and let $[U_t] \subset {}^*\mathcal{U}$ be a representing family of utilities, giving rise to the game $(T, V, E, [U_t]_{t \in T})$. Then for any coalition S, there is a finite number M_S such that if $[x(t)]_{t \in T}$ is a maximizing allocation for S, that is if $V(S) = \sum_{t \in S} U_t(x(t))$ and $[x(t)]_{t \in T}$ is strictly feasible for S, then $||x(t)|| \leq M_S$ for all $t \in S$.

Proof: The proof is given in Brown-Loeb, but it can be outlined very simply. If $x_j(t_0)$ is infinite for some $t_0 \in S$ and $j \leq \ell$, then clearly there must be an infinite number of traders in S, otherwise $[x(t)]_{t \in T}$ could not be feasible. And of those infinitely many traders, infinitely many must be assigned a finite amount of good j, $x_j(t)$ (otherwise $[x(t)]$ could not be feasible since the $I_j(t)$ are all finite). But then transfering to each such trader a noninfinitesimal amount of good j, Δ_j from $x_j(t_0)$, increases

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[1] Observe that we do not assume concavity of the utility functions.

[2] We shall never use assumption 3(d); "assumptions 1-3" means 1,2,3a,3b,3c.

$U_t(x(t))$ by a finite amount $U_t(x(t) + \Delta_j e_j) - U_t(x(t))$ and since there are an infinite number of such traders, increases the sum $\sum_{t \in S} U_t(x(t))$ by an infinite amount, whereas the loss to $U_{t_0}(x(t_0))$ can only be finite since U_{t_0} is bounded by a finite number (from assumption 3b). More rigorously, for all n let $R_n \subset S$ be the internal set $R_n = [t \in S \mid x_j(t) < n]$. Consider the ratio of elements in R_n to those in S, $\frac{|R_n|}{|S|}$. We would like to show that $\frac{|R_n|}{|S|} \not\approx 0$ for some finite n (that would imply that R_n contains an infinite number of traders). If $\frac{|R_n|}{|S|}$ is infinitesimal for all finite n, then in particular $\frac{|R_n|}{|S|} < 1/n$ for all $n \in N$, hence it must be that for some $r \in {}^*N \sim N$, $\frac{|R_r|}{|S|} < 1/r$, for if there were no such r then $[n \in {}^*N \mid \frac{|R_n|}{|S|} < 1/n]$ would be an internal set equal to the finite integers, which we already know is not internal. But if $\frac{|R_r|}{|S|} < 1/r$, then $\frac{|S \sim R_r|}{|S|} \simeq 1$, hence an infinite number $|S \sim R_r|$ have an infinite amount of good j and only a negligible part of S has a finite amount. That is a contradicton for we know that $\sum_{t \in S} x_j(t) = \sum_{t \in S} I_j(t)$ and all the $I_j(t)$ are finite. Thus after all there is a finite n such that $R_n = [t \in S \mid x_j(t) < n]$ is a nonnegligible part of S and therefore contains an infinite number of traders.

Now the result is evident, for take $\theta = \min(|R_n|, [x_j(t_0)])$ where $[x_j(t_0)]$ is the largest integer smaller than or equal to $x_j(t_0)$. Then let $R \subset R_n$ be any subset of R_n having θ elements, i.e. $|R| = \theta$. Then noting that $t_0 \notin R_n$, define

$$w(t) = \begin{cases} x(t) + e_j & \text{for } t \in B \\ x(t) - \theta e_j & \text{for } t = t_0 \\ x(t) & \text{otherwise} \end{cases}.$$

where e_j is the jth unit vector. Then $w(t)$ is strictly feasible on S, and $U_{t_0}(x(t_0)) - U_{t_0}(x(t_0)-\theta e_j)$ is finite since U_{t_0} is bounded while since $x(t)$ and $x(t) + e_j$ were both chosen finite, we know that $\frac{\partial U_t}{\partial X_j} \gtrsim 0$ for all $x\varepsilon[x(t),x(t) + e_j]$, hence $U_t(x(t) + e_j) - U_t(x(t)) \underset{\not\approx}{>} 0$ for all $t\varepsilon B$. But by construction B has an infinite number of elements, hence $\sum_{t\varepsilon S} U_t(w(t)) > \sum_{t\varepsilon S} U_t(x(t))$, contradicting the maximality of $(x(t))_{t\varepsilon S}$ QED

<u>Corollary</u>: Let $E = (t,I(t), >_t)$ be a nonstandard economy satisfying our assumptions and $\mathcal{U}$ a set of utilities also satisfying assumption 3. Then there exists a finite $M\varepsilon R$ such that for any representing family of utilities $[U_t]_{t\varepsilon T} \subset {}^*\mathcal{U}$ and game $(T,V,E,[U_t])$ and any coalition $S\subset T$, if $[x(t)]_{t\varepsilon S}$ is maximal for S with respect fo $[U_t]$, then $||x(t)|| \leq M$ for all $t\varepsilon S$.

<u>Proof</u>: Consider the internal set of integers A, where $n \varepsilon A$ if there exists a coalition S, a representing famlily of utilities $[U_t]_{t\varepsilon T} \subset {}^*\mathcal{U}$, a maximizing allocation $[x(t)]_{t\varepsilon S}$ for S with respect to $[U_t]_{t\varepsilon T}$ such that $\max_{t\varepsilon S} ||x(t)|| \geq n$. By our theorem, A contains no infinite integers, hence it is bounded from above and has a finite largest element M. QED

Thus our basic model involves the study of an infinite number of agents, but nevertheless we need only concern ourselves with assignments of goods lying in a compact set $F = [x\varepsilon R^{\ell}_{+} \mid ||x|| \leq M]$. In particular we know that $\nabla U_t(x) \geq a \underset{\not\approx}{>} 0$ for all $x\varepsilon F$ by assumption 3b. Moreoever, if $\frac{|S|}{w} \simeq 0$, then $\frac{1}{w} V(S) \simeq 0$ since each trader can have at most a finite utility. Furthermore, if $\frac{1}{w}|K| \simeq 0$ and $K\subset S$, then $\frac{1}{w}[V(S) - V(S\sim K)] \simeq 0$ (although it is trivial, we defer the proof of this to section 7).

It is easy to see how our nonstandard model makes concrete the intuitive hypothesis of perfect competition, that one trader by his own actions should

not be able to influence the terms of trade faced by any others. If $[x(t)]_{t\varepsilon T}$ is a competitive equilibrium for the nonstandard economy $E = (t, I(t), >_t)_{t\varepsilon T}$, then if trader t_0 disappears, the same assignment $[x(t)]_{t\varepsilon T\sim[t_0]}$ is feasible on $T\sim[t_0]$ and hence is still a competitive equilibrium for the economy $E' = (t, I(t), >_t)_{t\varepsilon T\sim[t_0]}$. Similarly the Core and Bargaining Sets of the games $(T,V,E,[U_t]_{t\varepsilon T})$ as well as the competitive payoff configurations and the natural payoff configurations are unaffected by the disappearance of a single trader. This is because they all rest on the notion of feasibility (rather than strict feasibility), which if $|T|$ is infinite and the initial endowments are more or less alike (all finite), is unaffected by the presence or absence of a single trader if $V(T) - V(T\sim[t])$ is finite for any t.

In order to prove that the value payoff configurations $[\phi_t]_{t\varepsilon T}$ are unaffected by the disappearance of one trader we shall need to demonstrate the important property that if S is any infinite coalition and $[x(t)]_{t\varepsilon S}$ is maximal for S, then there is a unique "price" vector p associated with $[x(t)]_{t\varepsilon S}$ (the Lagrangian of the maximization problem); moreover if $[y(t)]_{t\varepsilon S\sim[t_0]}$ is maximal for $S\sim[t_0]$ and gives rise to the "price" vector q, then $q \approx p$. After we give an argument in section 8 demonstrating this result, the Bargaining Set equivalence follows rather easily, as does the fact that the value payoff configurations are unaffected by the disappearance of a single trader.

Notice the difference between the nonstandard model and the continuum model. If a trader disappears in the nonstandard model we can calculate the difference $\sum_{t\varepsilon T} I(t) - \sum_{t\varepsilon T\sim[t_0]} I(t)$; the accounts in the nonstandard world would show a difference, but each agent would be so slightly affected that

he would not bother to alter his behavior. In the non-atomic model, on the other hand, the disappearance of a trader t_0 would not be revealed anywhere.

One consequence of assumptions 1-3 we should point out is that it is impossible to prove an analogous theorem to the finite representation theorem. Even if we assume the $>_t$ can be represented by concave utilities,[1] there are pareto optimal allocations in E for which there are no utilities $[U_t]_{t\varepsilon T} \subset {}^*\mathcal{U}$ such that $[x(t)]_{t\varepsilon T}$ gives rise to an imputation in the natural way. The reason is that the necessary weights λ_t might not be finite, contradicting the bounded differentiable assumptions on ${}^*\mathcal{U}$.

We close this section by showing that any nonstandard economic game $(T,V,E,[U_t]_{t\varepsilon T})$ satisfying assumptions 1-3 is essentially equivalent to a type economy where the number of distinct types v may be infinite but $v/w \approx 0$. This result plays no role in the rest of the paper.

<u>Type Economy Approximation Theorem</u>: If $E = (t,I(t),>_t)_{t\varepsilon T}$ is an economy and $[U_t]_{t\varepsilon T}$[2] a representing family of utilities satisfying assumptions 1-3, then we can find a type economy $\bar{E} = (t,\bar{I}(t),\bar{>}_t)_{t\varepsilon T}$ and a representing family of utilities $[\bar{U}_t]_{t\varepsilon T} \subset {}^*\mathcal{U}$ in which each trader is one of v types, where $v/w \approx 0$, and such that for all $t\varepsilon T$, $I(t) \approx \bar{I}(t)$ and if $[x(t)]_{t\varepsilon S}$ is maximal for S in $(T,V,E,[U_t]_{t\varepsilon T})$ or in $(T,\bar{V},\bar{E},[\bar{U}_t]_{t\varepsilon T})$, then $U_t(x(t)) \approx \bar{U}_t(x(t))$ for all $t\varepsilon S$. Moreoever, the Core, Bargaining Set, competitive payoff configurations, and value payoff configurations for $(T,V,E,[U_t]_{t\varepsilon T})$ are exactly the same as those for $(T,\bar{V},\bar{E},[\bar{U}_t]_{t\varepsilon T})$, hence proving the equivalence of those solution concepts in all nonstandard type economies $(T,\bar{V},\bar{E},[\bar{U}_t]_{t\varepsilon T})$ immediately implies their equivalence in all

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[1] That is, even with assumption 3d

[2] $[U_t]_{t\varepsilon T} \subset {}^*\mathcal{U}$

nonstandard economies satisfying assumptions 1-3. Thus the sequences of (unequally) replicated economies of Scarf-Debreu and Shapley are in a sense the most general sequences of economies for which these equivalences can be proved. On the other hand, it turns out that the proofs in $(T,V,E,[U_t]_{t\varepsilon T})$ directly are at least as simple as those which attempt to exploit the type economy nature of the games $(T,\bar{V},\bar{E},[\bar{U}_t]_{t\varepsilon T})$.

Proof: Consider the very large economic game $(\tilde{T},\tilde{V},\tilde{E},[\tilde{U}_t]_{t\varepsilon\tilde{T}})$ obtained from $(T,V,E,[U_t]_{t\varepsilon T})$ by replicating each trader in T $w = |T|$ times. Thus $\tilde{T}$ contains w copies of T and w^2 traders altogether. Observe, however, that $(\tilde{T},\tilde{V},\tilde{E},[\tilde{U}_t]_{t\varepsilon\tilde{T}})$ is a nonstandard game satisfying all of our assumptions 1-3, hence by proposition 1 there is some finite M such that for any $\tilde{S}\subset\tilde{T}$, if $[\tilde{x}(t)]_{t\varepsilon\tilde{S}}$ is maximal for $\tilde{S}$ with respect to $[\tilde{U}_t]_{t\varepsilon\tilde{T}}$, then $||\tilde{x}(t)|| < M$ for all $t\varepsilon S$. But now consider the standard space of continuous functions $C(\bar{M})$ defined over the compact set $\bar{M} = [x\varepsilon R^{\ell}_{+} \mid ||x|| \leq M]$. We can turn $C(\bar{M})$ itself into a compact Banach space by defining the usual sup norm $||f-g|| = \underset{x\varepsilon\bar{M}}{\mathrm{Max}} |f(x)-g(x)|$. Then since $C(\bar{M})$ is compact, for any standard $\varepsilon > 0$, it can be covered by a finite number of open balls of the form $f_{\varepsilon} = [g\varepsilon\ (\bar{M}) \mid ||f-g|| < \varepsilon]$. Hence if we consider the nonstandard extension $*C(\bar{M})$ of $C(\bar{M})$, then, for any standard $\varepsilon \gtrsim 0$, we can find a standard finite number of open balls (indeed simply the same balls as before) covering $*C(\bar{M})$. But then it follows that there must be some infinitesimal $\varepsilon > 0$, $\varepsilon \simeq 0$, such that $^*C(\bar{M})$ can be covered by v balls, of the form f_{ε}, where $v/w \simeq 0$. This follows easily from the fact that for any $n \gtrsim 0$, we can find a finite number of balls $\mu(n)$ (which implies $\mu(n)/w\simeq 0$) of radius less than $1/n$ that cover $*C(\bar{M})$, hence there must be at least one infinite integer α such that $\mu(\alpha)/w \simeq 0$ and $\mu(\alpha)$ balls of radius $1/\alpha$ cover $*C(\bar{M})$. This is another

example of the prolongation theorem. The set $[n\varepsilon{}^*N] \mid \mu(n)/w < 1/n]$ contains all the standard integers and it is internal, hence it must contain an infinite integer α .

We know that the $I(t)$ are also uniformly bounded by some finite positive number which we also take to be M. Then the space $[0,M]^{\ell}$ is compact and by exactly the same logic as before ${}^*[0,M]^{\ell}$ can be covered by μ infinitesimal open balls of radius $1/\beta \approx 0$, where $\mu(\beta)\mu(\alpha)/w \approx 0$. So consider the space ${}^*\mathcal{C}(\bar{M}) \times {}^*[0,M]^{\ell}$. It is covered by $\mu(\alpha)\mu(\beta)$ open sets of the form $f_{\alpha} \times B(x,\beta)$ where f_{α} is an open ball of the form $[g\varepsilon{}^*C(\bar{M}) \mid \|f-g\| < \alpha]$ and $B(x,\beta) = [y\varepsilon{}^*[0,M]^{\ell} \mid \|y-x\| < \beta]$. Let $v = \mu(\alpha)\mu(\beta)$. Then $v/w \approx 0$ by construction.

Now, every trader in $(T,V,E,[U_t]_{t\varepsilon T})$ is represented by an initial endowment $I(t)$ and a utility function $U_t \varepsilon{}^*\mathcal{C}^1 \subset \mathcal{C}(\bar{M})$. Hence each trader (U_t,I_t) "belongs" to one of the v open sets in ${}^*\mathcal{C}(\bar{M}) \times {}^*[0,M]^{\ell}$. Some traders may belong to more than one open ball if those open balls overlap - in that case choose one of the open balls and associate only that ball with the trader. For each of the v open sets that contains a trader, choose one trader $(U_t,I(t))$ contained in that open set as a representative. We construct $(T,\bar{V},\bar{E},[\bar{U}_t]_{t\varepsilon T})$ in the obvious way. For every $t\varepsilon T$, find the unique open set in our collection associated with $(U_t,I(t))$. Then define $(\bar{U}_t,\bar{I}(t))$ as the representative of the unique open set associated with $(U_t,I(t))$. In this way the game $(T,\bar{V},\bar{E},[\bar{U}_t]_{t\varepsilon T})$ is defined. Observe carefully that there are at most v distinct types of traders in $(T,\bar{V},\bar{E},[\bar{U}_t]_{t\varepsilon T})$ and that both $(T,V,E,[U_t]_{t\varepsilon T})$ and $(T,\bar{V},\bar{E},[\bar{U}_t]_{t\varepsilon T})$ can be considered restrictions of the bigger game $(\tilde{T},\tilde{V},\tilde{E},[\tilde{U}_t]_{t\varepsilon\tilde{T}})$. Hence the upper bound M we chose for $(\tilde{T},\tilde{V},\tilde{E},[\tilde{U}_t]_{t\varepsilon\tilde{T}})$ applies to the original game and the type economy game as well. From this it follows that for any $S\subset T$, $\frac{1}{|S|} V(S) \approx \frac{1}{|S|} \bar{V}(S)$.

To see this, let $[x(t)]_{t\in S}$ be maximal for S in $(T,V,E,[U_t]_{t\in T})$. Then $\frac{1}{|S|}\sum_{t\in S} x(t) = \frac{1}{|S|}\sum_{t\in S} I(t)$ and $||x(t)|| < M$ for all $t\in S$. But $\bar{I}(t) \approx I(t)$ for all $t\in S$, hence $\frac{1}{|S|}\sum_{t\in S} x(t) \approx \frac{1}{|S|}\sum_{t\in S} \bar{I}(t)$. Define $S_j = [t\in S \,|\, x_j(t) > \frac{1}{2}\frac{1}{|S|}\sum_{t\in S} I_j(t)]$. Since by assumption $I_j(t) \gtrapprox 0$ for all $t\in S$ and $I_j(t)$ is finite for all $t\in S$, $\frac{|S_j|}{|S|} \gtrapprox 0$ (otherwise we couldn't have $\sum_{t\in S} x_j(t) = \sum_{t\in S} I_j(t)$). Hence we can subtract the infinitesimal amount $\frac{1}{|S_j|}(\sum_{t\in S} I_j(t) - \sum_{t\in S} \bar{I}_j(t))$ from each $x_j(t)$ with $t\in S_j$. Repeating this process for $j = 1\ldots,\ell$, we get a feasible assignment $[\bar{x}(t)]_{t\in S}$ such that $\bar{x}(t) \approx x(t)$ for all $t\in S$ and $\sum_{t\in S} \bar{x}(t) = \sum_{t\in S} \bar{I}(t)$. So $\bar{U}_t(\bar{x}(t)) \approx \bar{U}_t(x(t))$ since $\bar{x}(t)$ and $x(t)$ are infinitesimally close and moreover $\bar{U}_t(x(t)) \approx U_t(x(t))$ by construction, hence $\frac{1}{|S|}\sum_{t\in S} U_t(x(t)) \approx \frac{1}{|S|}\sum \bar{U}_t(\bar{x}(t))$, and so $\frac{1}{|S|}\bar{V}(S) \approx \frac{1}{|S|}\bar{V}(S)$ for all $S\subset T$.

This immediately implies that the Core and Bargaining Sets of $(T,V,E,[U_t]_{t\in T})$ are exactly the same as the Core and Bargaining Sets of $(T,\bar{V},\bar{E},[\bar{U}_t]_{t\in T})$ respectively. To prove that the competitive equilibria of $(T,V,E,[U_t]_{t\in T})$ and $(T,\bar{V},\bar{E},[\bar{U}_t]_{t\in T})$ are the same requires us to take M big enough so that for every equilibrium and for each trader t the budget set of trader at the equilibrium prices is contained in $^*[0,M]^{\ell}$. This is not a problem once we have proved that all the equilibrium prices p must satisfy $0 \lessapprox p_i/p_j < K$ for some finite K. Rather than do this now and rather than prove the much harder result that the value payoff configurations of $(T,V,E,[U_t]_{t\in T})$ are the same as those of $(T,\bar{V},\bar{E},[\bar{U}_t]_{t\in T})$, we can observe that Brown-Robinson 1971 and Brown-Loeb 1976 and this paper show

that for any nonstandard economic game satisfying assumptions 1-3, the competitive equilibria, the core, the value payoff configurations, and the Bargaining Set are the same, hence the fact that the core of $(T,V,E,[U_t]_{t\varepsilon T})$ is the same as the core of $(T,\bar{V},\bar{E},[\bar{U}_t]_{t\varepsilon T})$ implies that the other corresponding solution concepts are equal as well. Of course if we wanted to use the type economy to prove the equivalence between all four solution concepts in an arbitrary game $(T,V,E,[U_t]_{t\varepsilon T})$ we would first have to prove the congruence of the value payoff configurations of $(T,V,E,[U_t]_{t\varepsilon T})$ with those of $(T,\bar{V},\bar{E},[\bar{U}_t]_{t\varepsilon T})$ without appealing to the equivalence of the four solution concepts. Proving the former, however, will nearly involve all the work required in proving the latter. Since it is only this latter equivalence we are interested in, it is best to proceed directly without recourse to the type economy approximation. QED

SECTION VII

Which is the Appropriate Setting

We attempt to argue here that of the three models embodying the hypothesis of perfect competition, the nonstandard model is the most suitable. All three approaches capture the spirit of perfect competition, namely that the actions of one individual should not affect market prices and hence not affect the market decisions of any of the other traders, although in the replicated sequence model this is asymptotically (never literally) true.

Though we are really interested in what happens in finite economies as the number of agents gets very large, the advantages that both the nonatomic model and the nonstandard model enjoy as against the replicated model (aside from the literal truth of the competitive hypothesis) are twofold: First, they give cleaner results since an equivalence rather than an ε convergence result can be demonstrated (although as Aumann himself points out, in the nonatomic case this is often more appearance than reality; for instance the nonatomic value is defined as the limit of values of finite games). Second, they are more general since it is usually a trivial matter, as Hildenbrand has demonstrated, to derive results for certain growing finite economies, including the replicated case, from the nonstandard or the continuum result. Again this may be a little misleading, for we showed in section 6 using nonstandard analysis that we often need to make so many assumptions on the boundedness of the characteristics of the traders that even in the infinite models we have in effect a type economy, with two big differences: there may be an infinite number v of types, though we still must have $\frac{v}{|T|} \approx 0$, and there need not be an equal number of traders of each type.

We claim that there are three reasons why it may be advantageous to use the nonstandard model rather than the nonatomic model. First, the nonstandard model avoids the paradox of a phantom trader who must not have any influence but still exist (he is allowed an infinitesimal influence on prices). In the continuum model it is impossible to speak of an individual trader, unless one means perhaps a nebulous dt. Second, in proving results about nonstandard economies one can combine the ideas usually found in replicated economies with those associated only with nonatomic economies. Typically convergence theorems for sequences of finite economies depend upon some sort of combinatorial property, like the law of large numbers, while theorems in nonatomic models often are verified by an appeal to Lyapunov's convexity theorem. In nonstandard analysis it is possible to prove an analogue of Lyapunov's theorem, and at the same time the traders are discrete, hence combinatorial principles like the law of large numbers also prove useful.

The third and most important reason for sometimes adopting, or at least investigating, the nonstandard model is its great simplicity. We have included a self-contained account of all the mathematical theory necessary to critically read this paper and many of the others in the economics literature that depend on nonstandard analysis.* The theorems which hold for the measure theoretic model and the nonstandard economic model are completely analogous, but the mathematical sophistication

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*This introduction to nonstandard analysis for the economist is a task others, more expert than me, should perhaps have undertaken. But since they haven't, I thought I would try.

required to prove them, far from being overwhelming, is much lower for the nonstandard model than the measure theoretic model. For instance, to prove that for a coalition S the maximum $V(S) = \max \{\int_S U_t(x(t))d\mu$ s.t. $x(t)\varepsilon R_+^{\ell}$ and $\int_S x(t)d\mu \leqq \int_S I(t)d\mu\}$ is attained is a hard problem in measure theory, whereas the analogous property, that for a coalition S the maximum

$$v(S) = \max [\sum_{t\varepsilon S} U_t(x(t)) \text{ s.t. } x(t)\varepsilon {}^*R_+^{\ell}, \sum_{t\varepsilon S} x(t) \leqq \sum_{t\varepsilon S} I(t)]$$

is attained is a triviality in the nonstandard model. Finding the maximum for $S\subset T$ of $[v(S) - \sum_{t\varepsilon S} x_t]$ is also trivial for the nonstandard model, as is showing the existence and uniqueness of the value payoff configuration, whereas in the atomless model these are problematical. It is true that until now all the analogous properties which hold for nonatomic economies and nonstandard economies have been proved first for the nonatomic economies, but this is only evidence of the imagination and skill of the measure theoreticians.

SECTION VIII

The Argument

We shall show that in any nonstandard game $(T,V,E,[U_t]_{t\varepsilon T})$ satisfying assumptions 1-3, the Bargaining Set and the Core coincide. Moreover, for general sequences of ever larger finite games $(T_n,V_n,E_n[U^n_t]_{t\varepsilon T_n})$, the Bargaining Set converges to the core.

Recall that the δ-Bargaining Set of a nonstandard game (T,V) is defined as the set of all payoff configurations (they need not be imputations or satisfy any other pareto optimal criterion) $[x_t]_{t\varepsilon T}$, $\frac{1}{w}\sum_{t\varepsilon T} x_t \leq \frac{1}{w} V(T)$, such that to any objection $(S,[z_t]_{t\varepsilon S},K)$ satisfying

(1) $K\subset S\subset T$, S nonnegligible

(2) $\frac{|K|}{w} \leq \delta$

(3) $\frac{1}{|S|}\sum_{t\varepsilon S} z_t \leq \frac{1}{|S|} V(S)$

(4) $z_t \gtrapprox x_t$ for all $t\varepsilon S$

there exists a counterobjection $(R,[w_t]_{t\varepsilon R})$ satisfying:

(1) $R\subset T$, $R\cap K = \emptyset$

(2) $\frac{1}{|R|}\sum_{t\varepsilon R} w_t \leq \frac{1}{|R|} V(R)$

(3) $w_t \geq z_t$ for all $t\varepsilon S\cap R$, $w_t \geq x_t$ for all $t\varepsilon R\sim S$.

The Bargaining Set of (T,V) is defined as the union of all the δ-Bargaining Sets for all noninfinitesimal δ, $\mathcal{B}S = \bigcup_{\delta \not\approx 0} \mathcal{B}S_\delta$.

As we have pointed out earlier, a natural method of proof is to show that infinitesimal perturbations of the utilities and initial endowments do not affect the Core or the Bargaining Set (or for that matter the competitive payoff

configurations), then to show that any nonstandard game $(T,V,E,[U_t]_{t\varepsilon T})$ can be infinitesimally perturbed to give a "type" economy game $(T,V,\bar{E},[\bar{U}_t]_{t\varepsilon T})$ where each trader in $\bar{E}$ is one of v types, where v may be infinite but $\frac{v}{w} \simeq 0$. We could conclude the proof rather easily by showing that all the solution concepts are equivalent in games $(T,V,\bar{E},[\bar{U}_t]_{t\varepsilon T})$ arising from type economies, if the ratio of types to agents is infinitesimal. In a preliminary version of this paper this was the approach adopted.

Here, however, we give a more direct proof, showing that the Core and Bargaining Set are identical in any game (T,V) in which the following two conditions hold:

(1) For any coalition S, $V(S) - V(S^{\sim}[t])$ is finite, hence $\frac{1}{w}|S| \simeq 0$ implies $\frac{1}{w}V(S) \simeq 0$. For any coalition S with $|S| \varepsilon {}^*N^{\sim}N$, $\frac{1}{|S|} \sum_{t\varepsilon S} [V(S)-V(S^{\sim}[t])] \simeq \frac{1}{|S|}V(S$

(2) $\frac{1}{|R|} V(R) \leq \frac{1}{|R|} \sum_{t\varepsilon R\cap S} [V(S) - V(S^{\sim}[t])] + \frac{1}{|R|} \sum_{t\varepsilon R^{\sim}S} [V(S\cup[t]) - V(S)]$ for any coalition R and any coalition S with $|S| \varepsilon {}^*N^{\sim} N$.

These two conditions hold for any game $(T,V,E,[U_t]_{t\varepsilon T})$ arising from a large economy with bounded differentiable utilities. Indeed, condition (1) is simply a restatement of the value equivalence theorem, and condition (2), which suggests that for large economies the function V is in some sense a concave constant returns to scale function (the convexifying effect of large numbers analogous to Lyapunov's theorem for measure spaces), is very easy to demonstrate once condition (1) holds. Condition (1) allows us to raise the amusing point that if the payoff configuration $[x_t]_{t\varepsilon T}$ is not in the core of (T,V), then we can construct an objection $(S,[y_t]_{t\varepsilon S},K)$ where $[y_t]_{t\varepsilon S}$ is nearly a value payoff configuration for the restricted game (S,V).

We repeat that the fundamental idea behind all of these equivalence proofs is that if $[x(t)]_{t\varepsilon S}$ is a maximal allocation with respect to the utilities $[U_t]_{t\varepsilon T}$ for a nonnegligible coalition S, giving rise to Lagrange multipliers

p , then perturbing the utilities infinitesimally or even adding a finite number of agents to S will result in a new maximal allocation $[y(t)]_{t\epsilon S}$ and Lagrange multipliers q such that q is infinitesimally close to p .

Our proof is divided into two parts. In the nontechnical first part we show that the Core and Bargaining Set coincide for any nonstandard game (T,V) satisfying conditions (1) and (2). In the second part of the proof we demonstrate that conditions (1) and (2) hold for any nonstandard economic game $(T,V,E,[U_t]_{t\epsilon T})$ satisfying assumptions 1-3. In fact conditions (1) and (2) are rather easy consequences of the value equivalence theorem proved by Brown-Loeb. We report their part of the argument for completeness and to make clear the connection between the Value equivalence and the Bargaining Set equivalence.

Part I

Proof of Theorem that: In any nonstandard game (T,V) where conditions (1) and (2) hold, the Core coincides with the Bargaining Set. In particular, the Bargaining Set payoff configurations are imputations (pareto optimal).

It is clear that the core of any nonstandard economy is contained in the Bargaining Set, for if $x = [x_t]_{t\epsilon T}$ is a payoff configuration in the core of (T,V) then by definition there can be no objection to x .

Consider a payoff configuration $[x_t]_{t\epsilon T} \equiv x$ such that $\frac{1}{w}\sum_{t\epsilon T} x_t \lesssim V(T)$ and assume that x is not in the core of (T,V) . We shall show that if (T,V) satisfies conditions (1) and (2), then x is not in the Bargaining Set of (T,V) either. We shall show it is not in the Bargaining Set of (T,V) by constructing a justified objection (an objection with no counterobjection).

The idea of the proof is straightforword. If x is not in the core of (T,V) , then $\operatorname{Max}_{S\subset T} \frac{1}{w}(V(S)-x(S)) = D \not\approx 0$ where $x(S) \equiv \sum_{t\epsilon S} x_t$. A natural idea

is to try to construct a justified objection with the coalition S_0 that achieves the maximum excess D (by transfer from the finite case the maximum is achieved).[1] Observe that S_0 is nonnegligible (hence $|S_0| \epsilon {}^*N\tilde{}N$), for otherwise we would have that

$\frac{1}{w} V(S_0) \simeq 0$, contradicting $\frac{1}{w} V(S_0) - \frac{1}{w} x(S_0) \underset{\not\simeq}{>} 0$.

From the maximizing property of S_0 it follows that $y_t \equiv V(S_0) - V(S_0 \tilde{}[t]) \geqq x_t$ for all $t \epsilon S_0$ (otherwise the coalition $(S_0 \tilde{}[t])$ would yield a greater excess $V(S) - x(S)$). Similarly $y_t \equiv [V(S_0 \cup [t]) - V(S_0)] \leqq x_t$ for any $t \notin S_0$ (otherwise $S_0 \cup [t]$ would yield a greater excess). But now we use the value equivalence theorem (condition (1)) to assert that the payoff $[y_t]_{t\epsilon S}$ is a value payoff configuration for the game (S_0, V), hence it is feasible for S_0; $\frac{1}{|S_0|} \sum_{t\epsilon S_0} y_t \equiv \frac{1}{S_0} \sum_{t\epsilon S_0} [V(S_0) - V(S_0 \tilde{}[t])] \simeq \frac{1}{|S_0|} V(S_0)$.

We are now ready to construct our justified objection $(S_0, [z_t]_{t\epsilon S_0}, K)$ which has the amusing property that $[z_t]_{t\epsilon S_0}$ is almost a value payoff configuration for the game (S_0, V).

Choose $K \subset S_0$ such that $0 \underset{\not\simeq}{<} \frac{|K|}{w} \leqq \delta$ so that $y_t - x_t \underset{\not\simeq}{>} 0$ for all $t \epsilon K$. This must be possible, for $\frac{1}{w} y(S_0) \simeq \frac{1}{w} V(S_0)$ and $\frac{1}{w}[V(S_0) - x(S_0)] \underset{\not\simeq}{>} 0$. Hence if we order the traders in S_0 according to which has the highest difference $y_t - x_t$ and take one by one those from the top of the list, so that the internal set K_α contains the α traders in S_0 with the highest excesses $y_t - x_t$, then for some α (with $\frac{\alpha}{w} \underset{\not\simeq}{>} 0$) we must have $y_t - x_t \underset{\not\simeq}{>} 0$ for all $t \epsilon K_\alpha$, and $0 \underset{\not\simeq}{<} \frac{|K_\alpha|}{w} \leq \delta$. This is true because if for some β, $\frac{\beta}{w} \simeq 0$ and already $y_{t_\beta} - x_{t_\beta} \simeq 0$, then $\frac{1}{w}[y(S_0) - x(S_0)] = \frac{1}{w}[y(S_0 \tilde{}K_\beta) - x(S_0 \tilde{}K_\beta)] + \frac{1}{w}[y(K_\beta) - x(K_\beta)] \leq \frac{1}{w} y(K_\beta) \simeq 0$ (by condition (1), y_t is finite for all t) contradicting the fact that $\frac{1}{w}[y(S_0) - x(S_0)] \simeq D \underset{\not\simeq}{>} 0$. Hence for some K_α, $0 \underset{\not\simeq}{<} \frac{\alpha}{w} = \frac{|K_\alpha|}{w}$ and $y_t \underset{\not\simeq}{>} x_t$ for all $t \epsilon K_\alpha$.

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[1] It will turn out best to choose $\bar{S}$ to maximize $\frac{1}{w}(V(S) - x(S) + \epsilon|S|)$ where ϵ is a fixed small standard positive number.

Let $K = K_\alpha$ and $\Delta = y(K) - x(K)$. Note that $\frac{\Delta}{w} \geq 0$. Define

$$z_t = \begin{cases} x_t + \frac{1}{2}(y_t - x_t) & \text{if } t\varepsilon K \\ y_t + \frac{1}{2}\frac{\Delta}{|S_0{\sim}K|} & \text{if } t\varepsilon S_0{\sim}K \end{cases}$$

Then $z_t \underset{\neq}{>} x_t$ for all $t\varepsilon S_0$ and $\sum\limits_{t\varepsilon S_0} z_t = \sum\limits_{t\varepsilon K} z_t + \sum\limits_{t\varepsilon S_0{\sim}K} z_t =$

$$\sum_{t\varepsilon K} x_t + \frac{1}{2}\Delta + \sum_{t\varepsilon S{\sim}K} y_t + \frac{1}{2}\Delta = x(K) + \Delta + y(S_0{\sim}K) = y(K) + y(S_0{\sim}K) =$$

$y(S_0)$ and $\frac{1}{|S_0|} y(S_0) \simeq \frac{1}{|S_0|} V(S_0)$. We could perturb our $[z_t]_{t\varepsilon S_0}$ each by an infinitesimal $\left(\frac{y(S_0)-V(S_0)}{|S_0|}\right)$ to get a strictly feasible $[z_t]_{t\varepsilon S_0}$ objection, if we had defined the Bargaining Set with strict feasibility.

We now proceed to show that there can be almost no counterobjecting coalition R, and this leads us to find another objections $[\bar{S},[z_t]_{t\varepsilon T},\bar{K}]$ for which there is no counterobjection at all.

Now from condition (2) for any $R \subset T$, $\frac{1}{|R|} V(R) \le \frac{1}{|R|} \sum_{t \in R} y_t$. But if R is a counterobjecting coalition, by definition $\frac{1}{|R|} \sum_{t \in R \cap S_o} z_t + \frac{1}{|R|} \sum_{t \in R \sim S_o} x_t \le \frac{1}{|R|} V(R)$. But $\frac{1}{|R|} \sum_{t \in R \cap S_0} z_t + \frac{1}{|R|} \sum_{t \in R \sim S_0} x_t > \frac{1}{|R|} \sum_{t \in R \cap S_0} y_t + \frac{1}{|R|} \sum_{t \in R \sim S_0} y_t =$

$\frac{1}{|R|} y(R) \ge \frac{1}{|R|} V(R)$ so if with a little more precision we could make the first inequality $>$ a stronger $\gneq$ inequality, we would have proved that there can be no counterobjection to the objection $(S_0, [z_t]_{t \in S}, K)$, hence for any $\delta \gneq 0$ the δ-Bargaining Set is equal to the core, hence the Bargaining Set, which is just the union of all the noninfinitesimal δ-Bargaining Sets, is itself equal to the Core.

In fact, this last step is easy to patch up; we only need to modify slightly our strategy. Take $\varepsilon \equiv \frac{1}{4} \delta D \gneq 0$, and define the new payoff configuration $\bar{x}_t = x_t - \varepsilon$. Then $\bar{x}(S) = x(S) - \varepsilon |S|$ for all $S \subset T$.

We can now proceed exactly as we did before, except using $[\bar{x}_t]_{t \in T}$ instead of $[x_t]$. Let $\bar{S}$ maximize $\frac{1}{w} [V(S) - \bar{x}(S)]$. Clearly $V(\bar{S}) - \bar{x}(\bar{S}) \geqq D + \varepsilon \frac{|S_0|}{w}$. As before $y_t \equiv V(\bar{S}) - V(\bar{S} \sim [t]) \geqq \bar{x}_t$ for all $t \in \bar{S}$ and

$$y_t \equiv V(\bar{S} \cup [t]) - V(\bar{S}) \leqq \bar{x}_t \quad \text{for all } t \notin \bar{S}.$$

As before, take $\bar{K}$ to be the coalition in $\bar{S}$ of size $\frac{\bar{K}}{w} \leqq \delta$ that maximizes $y(\bar{K}) - \bar{x}(\bar{K})$. It must be that $\frac{1}{w} \sum_{t \in \bar{K}} (y_t - \bar{x}_t) \gtrsim \delta D = 4\varepsilon \gneq 0$.

Let

$$z_t = \begin{cases} \bar{x}_t + \frac{1}{2|\bar{K}|} (y(\bar{K}) - \bar{x}(\bar{K})) & \text{if } t \in \bar{K} \\ \\ y_t + \frac{1}{2} \frac{1}{|\bar{S} - \bar{K}|} (y(\bar{K}) - \bar{x}(\bar{K})) & \text{if } t \in \bar{S} \sim \bar{K} \end{cases}$$

As before, we can show that $\frac{1}{|\bar{S}|} z(\bar{S}) = \frac{1}{|\bar{S}|} V(\bar{S})$ and $z_t \gneq \bar{x}_t$ for all $t \in S$ and for any R such that $R \cap K = \emptyset$, $\frac{1}{|R|} V(R) \le \frac{1}{|R|} y(R) < \frac{1}{|R|} \bar{x}(R \sim S) + \frac{1}{|R|} z(R \cap S)$.

Observe now that $z_t = \bar{x}_t + \frac{1}{2}\frac{1}{|\bar{K}|}(y(\bar{K}) - x(\bar{K})) \geq \bar{x}_t + \frac{1}{2}\frac{\delta D}{|\bar{K}|} \gneq \bar{x}_t +$

$\frac{1}{2}\frac{\delta D}{w} = \bar{x}_t + \frac{1}{4}\frac{\delta D}{w} + \frac{1}{4}\frac{\delta D}{w} = x_t + \varepsilon$ for all $t \varepsilon \bar{K}$.

For $t \varepsilon S \sim \bar{K}$, $z_t = y_t + \frac{1}{2}\frac{1}{|S-\bar{K}|}(y(\bar{K}) - \bar{x}(\bar{K})) \geq y_t + \frac{1}{2}\frac{1}{|S-\bar{K}|}\delta D \gneq y_t +$

$\frac{1}{2}\frac{\delta D}{w} \gneq y_t + \varepsilon$

Now look at R again.

$$\frac{1}{|R|}V(R) \leq \frac{1}{|R|}y(R) \leqq \frac{1}{|R|}\bar{x}(R\sim S) + \frac{1}{|R|}y(R\cap S)$$

$$\leqq \frac{1}{|R|}(x(R\sim S) - \varepsilon|R\sim S|) + \frac{1}{|R|}(z(R\cap S) - \varepsilon|R\cap S|)$$

Thus $\frac{1}{|R|}V(R) + \varepsilon \leq \frac{1}{|R|}x(R\sim S) + \frac{1}{|R|}z(R\cap S)$ where $\varepsilon \gneq 0$.

So not only is no counterobjection strictly feasible or even feasible, but in fact every coalition R fails to be a counterobjection by at least $\frac{1}{2}\varepsilon$ per trader.

Part II

Proofs of conditions 1 and 2 for games $(T,V) = (T,V,E,[U_t]_{t\varepsilon T})$ arising from nonstandard economies satisfying assumptions 1-3. Recall that $[x(t)]_{t\varepsilon S}$ is maximal for S with respect to the utility functions $[U_t]_{t\varepsilon S} \varepsilon {}^*\mathcal{U}$ iff $x(t) \varepsilon {}^*R_+^\ell$ for all $t\varepsilon S$ and $\sum_{t\varepsilon S} x(t) = \sum_{t\varepsilon S} I(t)$ and $V(S) = \sum_{t\varepsilon S} U_t(x(t))$. Recall also from the corollary to proposition 1 that there exists a finite M such that for any S and any maximal $[x(t)]_{t\varepsilon S}$, $||x(t)|| \leqq M$, for all $t\varepsilon S$, and that by assumption 3b, on $[x\varepsilon {}^*R_+^\ell \mid ||x|| \leq M]$, $\nabla U(x) \geqq a \gg\!\!\!\!/\; 0$ for all $U \varepsilon {}^*\mathcal{U}$.

<u>Proposition 2</u>: Let $[U_t]_{t\varepsilon T}$ be a representing family of utilities.[1] Given a

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[1]We assume in all these propositions that $(T,V,E,[U_t]_{t\varepsilon T})$ satisfies assumptions 1,2,3a,b,c.

coalition S , if $[x(t)]_{t\varepsilon S}$ is maximal for S then there exists a finite price vector $p \underset{\neq}{>>} 0$ such that $\nabla_j U_t(x(t)) \leqq p_j$ for all $t\varepsilon T$ and if $x_j(t) > 0$, then $\nabla_j U_t(x(t)) = p_j$.

Proof: The existence of p follows by transfer from the finite case. Since for any t , $\|x(t)\| \lesssim M$ by proposition 1, it follows that $p \geqq \nabla U_t(x(t)) \geqq a \underset{\neq}{>>} 0$, hence $p \underset{\neq}{>>} 0$, and for each j , there is some $t\varepsilon S$ such that $p_j = \nabla_j U_t(x(t))$ which is finite. QED.

Proposition 3: In proposition 2, if the $[U_t]$ are concave, then for any $y\varepsilon {}^*R^{\ell}_+$, $U_t(y) \leqq U_t(x(t)) + p^t(y-x(t))$, that is by rearranging terms, $U_t(y) - p^t(y-I(t)) \leqq U_t(x(t)) - p^t(x(t)-I(t))$ for all $t\varepsilon S$. Even if the U_t are not concave, if $|S| \varepsilon {}^*N\tilde{\ }N$ then for any $y\varepsilon {}^*R^{\ell}_+$ and $t\varepsilon S$, $U_t(y) \lesssim U_t(x(t)) + p^t(y-x(t))$, hence $U_t(y) - p^t(y-I(t)) \lesssim U_t(x(t)) - p^t(x(t) - I(t))$. This is precisely the convexifying effect of a large number of traders that we discussed earlier.

Proof: If the U_t are concave then by definition $U_t(y) \leqq U_t(x(t)) + \nabla U_t(x(t))(y-x(t))$. But from proposition 2, $0 \underset{\neq}{<<} \nabla_j U_t(x(t)) = p_j$ unless $x_j(t) = 0$, in which case $y_j - x_j(t) \geqq 0$ and $\nabla_j U_t(x(t))(y_j-x_j(t)) \leqq p_j(y_j-x_j(t))$. Thus $\nabla U_t(x(t))(y-x(t)) \leqq p^t(y-x(t))$ and so $U_t(y) \leqq U_t(x(t) + p^t(y-x(t))$.

For the rest of the proposition, suppose $|S| \varepsilon {}^*N\tilde{\ }N$ and for some $t_0 \varepsilon S$, $y\varepsilon {}^*R^{\ell}_+$, $U_{t_0}(y) \underset{\not\approx}{>} U_{t_0}(x(t_0)) + p^t(y-x(t_0))$. We shall show $[x(t)]_{t\varepsilon S}$ could not have been maximal, since giving trader t_0 y instead of $x(t_0)$ increases his utility by a noninfinitesimal amount greater than $p^t(y-x(t))$, while we can spread the loss $y-x(t)$ to all of the other traders around so thinly that the total loss is $\simeq p^t(y-x(t))$ which contradicts the maximality of $[x(t)]_{t\varepsilon S}$.

Note that for $U_{t_0}(y) \gtrapprox U_{t_0}(x(t_0)) + p^t(y-x(t_0))$ to hold, we must have that y is finite since U_{t_0} is bounded and $p^t y$ would be infinite if y were infinite. Thus we have only a finite amount to spread around. For $1 \leqq j \leqq \ell$, let $S_j = [t\varepsilon S | \; x_j(t) \geqq \frac{1}{2} \frac{1}{|S|} \sum_{t\varepsilon S} I_j(t) \underset{\not\approx}{>} 0]$. Since the $x_j(t)$ are all finite by proposition 1 and the $I_j(t)$ are all finite and noninfinitesimal by assumption, S_j must contain an infinite number of traders (in fact $|S_j|/|S| \underset{\not\approx}{>} 0$) and by proposition 2, for each $t\varepsilon S_j$, $\nabla_j U_t(x(t)) = p_j$. Let $w_j(t) = 0$ if $t \notin S_j$ and $\frac{1}{|S_j|}(y_j - x_j(t_0)$ if $t\varepsilon S_j$. Note that $w_j(t) \simeq 0$,

hence for $t\varepsilon S_j$, letting $e_j = (0, 0, 1, 0\ldots, 0)$, $U_t(x(t) - w_j(t)e_j) =$ $U_t(x(t)) - \nabla_j U_t(x(t))w_j(t) + \varepsilon_j(t) = U_t(x(t)) - p_j w_j(t) + \varepsilon_j(t)$ where

$\frac{\varepsilon_j(t)}{w_j(t)} \simeq 0$. Let $w(t) = \sum_{j=1}^{\ell} w_j(t)\varepsilon_j$. Then $U_t(x(t)-w(t)) \simeq U_t(x(t)) -$

$p^t w(t) + \varepsilon(t)$ where $\frac{\varepsilon(t)}{\|w(t)\|} \approx 0$. Thus the strictly feasible

assignment $[x(t)-w(t)]_{t\varepsilon S^\sim[t_0]} \cup [x(t_0) = y]$ satisfies $\sum_{t\varepsilon S^\sim[t_0]} U_t(x(t)-w(t)) +$ $U_{t_0}(y) = \sum_{t\varepsilon S^\sim[t_0]} [U_t(x(t)) - p^t w(t) + \varepsilon(t)] + U_{t_0}(y) = \sum_{t\varepsilon S^\sim[t_0]} U_t(x(t) -$

$p^t(y-x(t_0) + \sum_{t\varepsilon S^\sim[t_0]} \varepsilon(t) + U_{t_0}(y)$. Since $\sum_{t\varepsilon S^\sim[t_0]} \varepsilon(t) \leqq |S|\varepsilon$ where $\frac{\varepsilon}{\frac{1}{|S_j|}(y_j - x_j)} \simeq 0$

for all $j = 1\ldots, \ell$ it follows that $|S|\varepsilon \approx 0$ since $y_j - x_j$ is finite and $|S|/|S_j|$ is finite. Hence $\sum_{t\varepsilon S^\sim[t_0]} U_t(x(t)-w(t)) + U_{t_0}(y) \approx \sum_{t\varepsilon S^\sim[t_0]} U_t(x(t)) -$

$p^t(y-x(t_0)) + U_{t_0}(y) \underset{\not\approx}{>} \sum_{t\varepsilon S} U_t(x(t))$ contradicting the maximality of $[x(t)]_{t\varepsilon S}$. QED

<u>Proposition 4</u>: Let $[x(t)]_{t\varepsilon S}$ be a maximal allocation for S with respect to the representing family of utilities $[U_t]_{t\varepsilon T}$ and let $p \underset{\not\approx}{>>} 0$ be the vector associated with $[x(t)]_{t\varepsilon S}$ by proposition 2. Let $t_0 \varepsilon S$ and let $[y(t)]_{t\varepsilon S^\sim[t_0]}$ be maximal for the coalition $S^\sim[t_0]$ and give rise to the vector $q \underset{\not\approx}{>>} 0$ as

in proposition 2. If $|S|\varepsilon{*}N\tilde{}N$, then $p \simeq q$ and $V(S) - V(S\tilde{}[t_0]) \simeq U_{t_0}(x(t_0)) - p^t(x(t_0) - I(t_0))$.

<u>Proof</u>: Observe that $V(S) = \sum_{t\varepsilon S} U_t(x(t))$ and that after being eliminated, t_0 does not consume $x(t_0)$ nor contribute his endowment $I(t_0)$, hence by repeating the logic of the proof of proposition 3 we could show $V(S\tilde{}[t_0]) \gtrsim V(S) - U_{t_0}(x(t_0)) + p^t(x(t_0) - I(t_0))$ since we could divide up the $x(t_0) - I(t_0)$ among so many traders with gradients $\nabla_j U_t(x(t)) = p_j$ for $x_j(t) > 0$ that the total gain in utility would be $\approx p^t(x(t_0) - I(t_0))$. Hence $V(S) - V(S\tilde{}[t_0]) \lesssim U_{t_0}(x(t_0)) - p^t(x(t_0) - I(t_0))$.

From proposition 3 we have that for any y , in particular for $y = y(t)$ we have $U_t(y(t)) \lesssim U_t(x(t)) + p^t(y(t) - x(t))$ for all $t\varepsilon S$. Hence

$$\frac{1}{|S|}\sum_{t\varepsilon S\sim[t_0]} U_t(y(t) \lesssim \frac{1}{|S|}\sum_{t\varepsilon S\sim[t_0]} U_t(x(t)) + \frac{1}{|S|}\sum_{t\varepsilon S\sim[t_0]} p^t(y(t) - x(t)) =$$

$$\frac{1}{|S|}\sum_{t\varepsilon S\sim[t_0]} U_t(x(t)) + \frac{1}{|S|}(p^t(x(t_0) - I(t_0)))$$. But from the last paragraph we know that $\frac{1}{|S|}\sum_{t\varepsilon S\sim[t_0]} U_t(y(t)) = \frac{1}{|S|}V(S\tilde{}[t_0]) \gtrsim \frac{1}{|S|}[V(S) - U_{t_0}(x(t_0)) + p^t(x(t_0) - I(t_0))] \simeq \frac{1}{|S|}[\sum_{t\varepsilon S\tilde{}[t_0]} U_t(x(t)) + p^t(x(t_0) - I(t_0))]$. Thus we have that for all but a negligible number of $t\varepsilon S$, $U_t(y(t)) \simeq U_t(x(t)) + p^t(y(t) - x(t))$.

By using exactly the same logic we could have also shown that $V(S) \gtrsim V(S\tilde{}[t_0]) + U_{t_0}(x(t_0)) - q^t(x(t_0) - I(t_0))$, so that $V(S) - V(S\tilde{}[t_0]) \gtrsim U_{t_0}(x(t_0)) - q^t(x(t_0) - I(t_0))$, and for all but a negligible number of $t\varepsilon S$, $U_t(x(t)) \simeq U_t(y(t)) + q^t(x(t) - y(t))$.

Of course once we prove $p \simeq q$, we shall also have $V(S) - V(S\tilde{}[t_0]) \simeq U_{t_0}(x(t_0)) - p^t(x(t_0) - I(t_0))$.

Let j be given and choose a t such that $y_j(t) > 0$ and $U_t(y(t)) \simeq U_t(x(t)) + p^t(y(t) - x(t))$, so $U_t(x(t)) - U_t(y(t)) \simeq p^t(x(t) - y(t))$. Now, let $y \varepsilon {}^*R_+^\ell$ be arbitrary. Then

$$\begin{aligned} U_t(y) - U_t(y(t)) &= [U_t(y) - U_t(x(t))] + [U_t(x(t)) - U_t(y(t))] \\ &\simeq [U_t(y) - U_t(x(t))] + p^t(x(t) - y(t)) \\ &\lesssim p^t(y - x(t)) + p^t(x(t) - y(t)) \\ &= p^t(y - y(t)) . \end{aligned}$$

Hence for arbitrary y, $U_t(y) \lesssim U_t(y(t)) + p^t(y - y(t))$ and also $U_t(y) \lesssim U_t(y(t)) + q^t(y - y(t))$ where $y_j(t) > 0$. Thus it follows that $p_j \simeq \nabla_j U_t(y(t)) = q_j$. Thus we could show for all j that $p_j \simeq q_j$, hence $p \simeq q$. QED

<u>Corollary 1</u>: Let $(T,V,E,[U_t]_{t\varepsilon T})$ satisfy assumptions 1-3. Then for any coalition S and $t \varepsilon S$, $V(S) - V(S^\sim[t])$ is finite.

<u>Proof</u>: For S finite, $V(S) - V(S^\sim[t]) \lesssim V(S)$ which is itself finite. For $|S| \varepsilon {}^*N^\sim N$, we can apply proposition 4 to obtain $V(S) - V(S^\sim[t]) \simeq U_t(x(t)) - p^t(x(t) - I(t))$ where $x(t)$ is part of a maximal allocation for S and p is finite. By proposition 1 $x(t)$ is finite and by assumption $I(t)$ is finite. QED

<u>Corollary 2</u>: If $|S| \varepsilon {}^*N^\sim N$ and $K \subset S$ and $\frac{|K|}{|S|} \simeq 0$, then $\frac{1}{|S|} V(S) = \frac{1}{|S|} V(S^\sim K)$.

<u>Proof</u>: Let $[x(t)]_{t\varepsilon S}$, p be maximal for S and let $[y(t)]_{t\varepsilon S^\sim K}$, q be maximal for $S^\sim K$. Then by applying our usual methods we can show that $V(S^\sim K) \gtrsim V(S) - \sum_{t\varepsilon K} U_t(x(t)) + p^t(\sum_{t\varepsilon K} (x(t) - I(t)))$. Hence $\frac{1}{|S|}(V(S) - V(S^\sim K)) \lesssim \frac{1}{|S|}(\sum_{t\varepsilon K} U_t(x(t)) - p^t(\sum_{t\varepsilon K} (x(t) - I(t))) \simeq 0$. QED

<u>Corollary 3</u>: If $\frac{1}{w}|S| \simeq 0$, then $\frac{1}{w}V(S) \simeq 0$.

<u>Proof</u>: By assumption there is some finite W such that $U_t(x) \leqq W$ for all $x\varepsilon{}^*R_+^{\ell}$ and all $t\varepsilon T$. Hence $\frac{1}{w}V(S) = \frac{1}{w}\sum_{t\varepsilon S} U_t(x(t)) \leqq \frac{1}{w}|S|\ W \simeq 0$.

QED

For the sake of completeness we state the following corollary:

<u>Corollary 4</u>: Let $(T,V,E,[U_t]_{t\varepsilon T})$ satisfy assumptions 1-3. Then even if the utilities $[U_t]_{t\varepsilon T}$ are not concave, there exists a unique (up to infinitesimal perturbations) competitive payoff configuration.[1]

<u>Condition 1 Holds</u>: For any game $(T,V,E,[U_t])$ satisfying assumptions 1-3, and for any coalition S with $|S|\varepsilon{}^*N\tilde{\ }N$, $\frac{1}{|S|}V(S) \simeq \frac{1}{|S|}\sum_{t\varepsilon S}[V(S) - V(S\tilde{\ }[t])]$.

<u>Proof</u>: This follows immediately from proposition 4, since for some $[x(t)]_{t\varepsilon T}, p$
$V(S) - V(S\tilde{\ }[t]) \simeq U_t(x(t)) - p^t(x(t) - I(t))$ for all $t\varepsilon T$, hence

$$\frac{1}{|S|}[\sum_{t\varepsilon S}[V(S) - V(S\tilde{\ }[t])] \simeq \frac{1}{|S|}\sum_{t\varepsilon S}[U_t(x(t)) - p^t(x(t) - I(t))] = \frac{1}{|S|}\sum_{t\varepsilon S}U_t(x(t)) = \frac{1}{|S|}V(S) .$$

QED

<u>Condition 2 Holds</u>: Let S be a nonnegligible coalition, so $|S|\ \varepsilon\ {}^*N\tilde{\ }N$ in the game $(T,V,E,[U_t])$ and let R be any coalition in T . Then

$$\frac{1}{|R|}V(R) \leq \frac{1}{|R|}\sum_{t\varepsilon R\cap S}[V(S) - V(S\tilde{\ }[t])] + \frac{1}{|R|}\sum_{t\varepsilon R\tilde{\ }S}[V(S\cup[t]) - V(S)] .$$

<u>Proof</u>: Let $[y(t)]_{t\varepsilon R}, q$ be maximizing for R . We know that there is some $p \underset{\neq}{>>} 0$ and $[x(t)]_{t\varepsilon S}$ such that for all $t\varepsilon S$, $V(S) - V(S\tilde{\ }[t]) \simeq U_t(x(t)) - p^t(x(t) - I(t))$ where $[x(t)]_{t\varepsilon S}$ is a maximizing allocation for S . But for any $t_0 \notin S$, consider a maximizing $[\bar{x}(t)]_{t\varepsilon S\cup[t_0]}$ for $S\cup[t_0]$. Then it is also true that $[\bar{x}(t)]$ is feasible on S , hence with the same p , $V(S\cup[t_0]) - V(S) \simeq U_{t_0}(\bar{x}(t_0)) - p^t(\bar{x}(t_0) - I(t_0))$

1 The proof follows immediately from the fact that by proposition 4 p is unique.

Use the notation $x(t)$ for all $t \varepsilon S$ and $x(t_0) = \bar{x}(t_0)$ for $t_0 \notin S$.[1] Then we have an assignment $[x(t)]_{t\varepsilon T}$ such that restricted to S, $[x(t)]_{t\varepsilon S}$ is a maximizing allocation and $V(S) - V(S\sim[t]) \simeq U_t(x(t)) - p^t(x(t) - I(t))$ while for $t \notin S$, $V(S\cup[t]) - V(S) \simeq U_t(x(t)) - p^t(x(t) - I(t))$. Let $\delta_S = \underset{t\varepsilon S}{\mathrm{Max}}\, [U_t(x(t)) - p^t(x(t) - I(t)) - V(S) + V(S\sim[t])]$ and let $\delta_{\bar{S}} = \underset{t\notin S}{\mathrm{Max}}\, [U_t(x(t)) - p^t(x(t) - I(t)) - V(S\cup[t]) + V(S)]$. Then $\delta(S) = |\delta_S| + |\delta_{\bar{S}}|$ is infinitesimal. Observe further that each $x(t)$ was part of a maximizing allocation for some coalition with an infinite number of traders. Hence by proposition 3, for any $y \varepsilon {}^*R^{\ell}_{+}$, $U_t(y) - p^t(y - I(t)) \lesssim U_t(x(t)) - p^t(x(t) - I(t))$ for all $t\varepsilon T$. In particular, $U_t(y(t)) - p^t(y(t) - I(t)) \lesssim U_t(x(t)) - p^t(x(t) - I(t))$. But $V(R) = \sum_{t\varepsilon R} U_t(y(t)) = \sum_{t\varepsilon R} U_t(y(t)) - p^t \sum_{t\varepsilon R} (y(t) - I(t)) = \sum_{t\varepsilon R} [U_t(y(t)) - p^t(y(t) - I(t))]$.

Let $(S,R) = \underset{t\varepsilon R}{\mathrm{Max}}\, [U_t(y(t)) - p^t(y(t) - I(t)) - U_t(x(t)) + p^t(x(t) - I(t))]$.

Then (S,R) is infinitesimal, and $V(R) = \sum_{t\varepsilon R} [U_t(y(t) - p^t(y(t) - I(t))] \leqq \sum_{t\varepsilon R} [U_t(x(t)) - p^t(y(t) - I(t)) + (S,R)] \leqq \sum_{t\varepsilon R\cap S} [V(S) - V(S\sim[t]) + \delta_S + (S,R)] + \sum_{t\varepsilon R\sim S} [V(S\cup[t]) - V(S) + \delta_{\bar{S}} + (S,R)]$. Thus $V(R) \leqq \sum_{t\varepsilon R\cap S} [V(S) - V(S\sim[t])] + \sum_{t\varepsilon R\sim S} [V(S\cup[t]) - V(S)] + |R|\,(\delta(S) + (S,R))$ where $\delta(S)$ and (S,R) are infinitesimal. Hence $\frac{1}{|R|} V(R) \lesssim \frac{1}{|R|} \sum_{t\varepsilon R\cap S} [V(S) - V(S\sim[t])] + \frac{1}{|R|} \sum_{t\varepsilon R\sim S} [V(S\cup[t]) - V(S)]$. QED

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[1] Where for each $t_0 \notin S$ we find a (different) maximizing allocation $[\bar{x}(t)]_{t\varepsilon S\cup[t_0]}$.

SECTION IX

Limit Theorem

Recall that for a finite game (T,V) we define the δ-Bargaining Set δ-BS as any payoff configuration $[x_t]_{t\varepsilon T}$, $\sum_{t\varepsilon T} x_t \leqq V(T)$, such that to every objection $(S,[y_t]_{t\varepsilon S},K)$ satisfying

(1) $K\subset S\subset T$

(2) $\frac{|K|}{|T|} \leqq \delta$

(3) $\sum_{t\varepsilon S} y_t \leqq V(S)$

(4) $y_t > x_t$ for all $t\varepsilon S$

there exists a counterobjection $(R,[w_t]_{t\varepsilon R})$ satisfying:

(1) $R\subset T$, $R\cap K = \emptyset$

(2) $\sum_{t\varepsilon R} w_t \leqq V(R)$

(3) $w_t \geqq y_t$ for all $t\varepsilon S\cap R$

(4) $w_t \geqq x_t$ for all $t\varepsilon R\sim S$

Note that if we applied this definition to a nonstandard economy, it would still make sense; in fact it would define a set included in the nonstandard δ-Bargaining Set.

We can prove that for very general sequences of finte economies the δ-Bargaining Set converges to the core. Specifically, we do not require concavity of the utility functions (or convexity of the underlying preferences), we do not require a "replicated" sequence of economies, and finally the Bargaining Set is not restricted to imputations (pareto optimal payoffs). We prove nevertheless that in large economies the Bargaining Set payoff configurations are nearly in the core, hence nearly imputations as well.

<u>Limit Theorem</u>:

Let $(T_n, V_n E_n [U_t^n]_{t\varepsilon T_n})$ be a sequence of finite transferable utility games derived from the economies $E_n = (t, I^n(t), >_t^n)_{t\varepsilon T_n}$ satisfying the following properties:

(1) $|T_n| \to \infty$

(2) There is a vector $a >> 0$ and a vector $b >> 0$ such that for all n and all $t\varepsilon T_n$, $a \leqq I^n(t) \leqq b$.

(3) There is a set of utilities $\mathcal{U}$ satisfying the same Assumption 3 as before such that for all n and all $t\varepsilon T_n$, $U_t^n \varepsilon \mathcal{U}$.

Let $\delta > 0$ be given. Then for any $\varepsilon > 0$, there is an integer $N(\varepsilon)$ such that if $n \geqq N(\varepsilon)$ and $[x_t]_{t\varepsilon T^n}$ is a payoff configuration in the δ-Bargaining Set of T^n, then for any coalition $S \subset T_n$, $\frac{1}{|T_n|} V_n(S) \leqq \frac{1}{|T_n|} \sum_{t\varepsilon S} x_t + \varepsilon$. In particular, for n big enough, any payoff configuration in the δ-Bargaining Set of T^n is almost an imputation.

<u>Proof</u>: Let E_n be the function associating with every natural number n the economy E_n. Let V be the function associating with every natural number n the game $(T_n, V_n, E_n, [U_t^n]_{t\varepsilon T^n})$. Suppose there is some $\varepsilon > 0$ and a sequence of games $V(n_K)$ and payoff configurations $[x_t^{n_K}]_{t\varepsilon T^{n_K}}$ in the δ-Bargaining Set of $V(n_K)$ such that for some $S_{n_K} \subset T_{n_k}$),

$$\frac{1}{|T_{n_K}|} V(S_{n_K}) - \varepsilon > \frac{1}{|T_{n_K}|} \sum_{t\varepsilon S_{n_K}} x_t^{n_K}$$. Then consider the function

${}^*V_1(n_K) = (T_{n_K}, V_{n_K}, E_{n_K} [U_t^{n_K}]_{t\varepsilon T_{n_K}})$ which must be defined for nonstandard integers K, and the nonstandard economy $E(n_K)$ for any fixed infinite integer K. Call the game V and the economy E. Then V must satisfy all the assumptions 1-3 of a nonstandard exchange economy. By transfer there will a payoff configuration

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THE BARGAINING SET AND NONSTANDARD ANALYSIS.(U)

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$[x_t]_{t\epsilon T} \equiv [x_t^{n_K}]_{t\epsilon T_{n_K}}$ such that $[x_t]_{t\epsilon T}$ satisfies the conditions of the <u>finite</u> δ-Bargaining Set applied to (T,V) and such that for some $S \subset T$, $\frac{1}{|T|} V(S) - \varepsilon > \sum_{t\epsilon S} x_t$. Clearly this implies that $[x_t]_{t\epsilon T}$ is in the nonstandard δ-Bargaining Set, which is impossible since $\frac{1}{|T|} V(S) - \varepsilon > \sum_{t\epsilon S} x_t$ implies in particular that $\frac{1}{|T|} V(S) > \varepsilon \underset{\not\approx}{>} 0$ and so S must be a nonnegligible coalition. But this shows that $[x_t]_{t\epsilon T}$ is in the δ-Bargaining Set but not in the core of the nonstandard economic game $(T,V,E,[U_t]_{t\epsilon T})$, which contradicts our equivalence theorem. QED

SECTION X

Further Questions

We have sought to give a new definition of the Bargaining Set, closely related to the original spirit of the Aumann-Maschler definition, which is sensible in a large economy, that is in an economy which embodies the idealized hypothesis of perfect competition. We were led to replace the single leader of an objection, an idea not appropriate to the spirit of perfect competition or to the nonstandard model of a large economy and meaningless in the continuum model of a large economy, with an arbitrarily small but nonnegligible set of traders. With this change we were able to simplify the convergence proof discovered by Shapley for sequences of replicated economies, laying bare the relationship between the value equivalence and the Bargaining Set equivalence, and at the same time eliminating several restrictive assumptions including concavity of the utility functions, the special sequences of replicated type economies used by Shapley, and even the assumption of pareto optimality. On the other hand, several further questions can be raised which have not been answered in this paper. Presumably we could have applied our definition of the Bargaining Set to the continuum model; it might be interesting to check whether the usual equivalence holds there as well. We conjecture that it does; indeed if a direct proof is sought, the second part of the argument given in our nonstandard proof can be duplicated almost verbatim for the continuum model. The first part of the argument, which was trivial for the nonstandard model since it only involved exploring the elementary properties of a set S_0 maximizing $V(S) - x(S)$, requires more care in the continuum model; it is not even clear that the maximum $V(S) - x(S)$ is attained.

A second, and perhaps more difficult, problem is to prove that the Bargaining Set we have defined exists in any nonstandard (or nonatomic) game (T,V) even if it is not derived from an economic market. One of the most attractive properties of the old definition of the Bargaining Set is that for finite games it is always nonempty. We can at least check that for some simple nonatomic games, such as the voting game (T,V) defined by $V(S) = 1$ if $\mu(S) > 1/2$, $V(S) = 0$ if $\mu(S) \leq 1/2$ the core is empty but the Bargaining Set contains the payoff configuration $x_t = 1$ for all $t \in T$. In fact it contains all the payoff configurations of the form $x_t = C$ where C is a constant smaller than or equal to 1. Of course if the core of a game (T,V) is nonempty, the Bargaining Set must also be nonempty.

Several other questions also suggest themselves. We saw in our discussion of economic markets and game theory that certain market allocations in E (for instance the competitive equilibria or even all the pareto optimal) can be represented in a natural way in an appropriately chosen associated transferable utility game $(T,V,E,[U_t]_{t\in T})$. (By competitive payoff configurations or imputations.) Furthermore, the value allocations were defined as all those allocations $[x(t)]_{t\in T}$ in E for which we could find an associated game $(T,V,E,[U_t]_{t\in T})$ in which $[x(t)]_{t\in T}$ gives rise to a value payoff $[\phi_t]_{t\in T}$ in the natural way, $\phi_t = U_t(x(t))$ for all $t \in T$. The equivalence of the competitive equilibria and value allocations in E is proved by showing that in all of the transferable utility games $(T,V,E,[U_t])$ associated with E, the competitive payoff distribution and the value payoff are the same. We would like to define an ordinal Bargaining Set for an economy E and then show that for any large E the Bargaining Set and the Core are nearly the same. We could by analogy to the value allocation define the Bargaining Set

of an economy E as the set of all allocations $[x(t)]_{t\epsilon T}$,

$\sum_{t\epsilon T} x(t) \leqq \sum_{t\epsilon T} I(t)$, for which we can find an associated game $(T,V,E,[U_t]_{t\epsilon T})$ in which $[x(t)]_{t\epsilon T}$ gives rise to a Bargaining Set payoff $[x_t]_{t\epsilon T}$ in the natural way $x_t = U_t(x(t))$ for all $t\epsilon T$. Of course our transferable utility game equivalence theorem implies the equivalence of the core and this Bargaining Set for large E.

Furthermore, there seems to be no reason why we cannot define the Bargaining Set of a nonstandard economy directly: We might say that in a nonstandard economy $E = (t,I(t),>_t)_{t\epsilon T}$ the feasible assignment $[x(t)]_{t\epsilon T}$ is in the ordinal δ-Bargaining Set iff to every <u>objection</u> $(S,[y(t)]_{t\epsilon S},K)$ satisfying:

(1) $K \subset S \subset T$, S nonnegligible

(2) $\frac{|K|}{|T|} \leqq \delta$

(3) $\frac{1}{|S|}\sum_{t\epsilon S} y(t) \leq \frac{1}{|S|}\sum_{t\epsilon S} I(t)$

(4) $y(t) >>_t x(t)$, that is for all $y \sim y(t)$ and $x \sim x(t)$, $y >_t x$

there exists a <u>counterobjection</u> $(R,[w(t)]_{t\epsilon R})$ such that

(1) $R \subset T$ and $R \cap K = \emptyset$

(2) $\frac{1}{|R|}\sum_{t\epsilon R} w(t) \leq \frac{1}{|R|}\sum_{t\epsilon R} I(t)$

(3) $w(t) \gtrsim_t y(t)$ for all $t\epsilon R\cap S$

(4) $w(t) \gtrsim_t x(t)$ for all $t\epsilon R \sim S$

As before we could define the ordinal Bargaining Set as the union of all the δBargaining Sets of E with $\delta \not> 0$. The problem remains to show that this ordinal Bargaining Set is equivalent to the Core, or what

is the same thing, to show that this ordinal Bargaining Set agrees with the Bargaining Set in E derived from the associated transferable utility game Bargaining Sets. Observe that we could have given an analogous definition for finite games by replacing all the infinitesimal inequalities $\gtrsim$ and $\underset{\approx}{>}$ by the standard inequalities $\geqq$ and $>$ and by taking $\delta = 1/|T|$ so K contains one trader, the leader of the objection. Of course proving that this Bargaining Set is nonempty for all finite economies is non trivial (the core may be empty in a small economy). It may not be true.

We can proceed one step further and try to propose a definition of the Bargaining Set of <u>non</u>transferable games $(T,\vec{V})$ which would agree with the Bargaining Set just defined on economies E. For finite games $(T,\vec{V})$ this might take the form: $[x_t]_{t\varepsilon T}$ is in the Bargaining Set of the non-transferable game $(T,\vec{V})$ iff $[x_t]_{t\varepsilon T} \varepsilon \vec{V}(T)$ and to every objection $(S,[y_t]_{t\varepsilon S},t_0)$ satisfying

(1) $t_0 \varepsilon S\ T$

(2) $[y_t]_{t\varepsilon S} \varepsilon \vec{V}(S)$

(3) $y_t > x_t$ for all $t\varepsilon S$

there exists a counterobjection $(R,[w_t]_{t\varepsilon R})$ such that

(1) $R\subset T$ and $t_0 \notin R$

(2) $[w_t]_{t\varepsilon R} \varepsilon \vec{V}(R)$

(3) $w_t \geqq y_t$ for all $t\varepsilon R\cap S$

(4) $w_t \geqq x_t$ for all $t\varepsilon R\tilde{\ }S$

Again the question arises whether this notion of the Bargaining Set is always nonempty. It seems unlikely that the truth is so fortunate, but perhaps yet another definition can be proposed which does have this property.

APPENDIX

We give, in a leisurely fashion, the proof of the theorem of section 2.

Lemma 1: For any economy $E = (T, I(t), >_t)_{t\epsilon T}$ and representing family of utilities $[U_t]_{t\epsilon T} \subset \mathcal{U}$ satisfying assumption 1-3, the game $(T, V, E, [U_t]_{t\epsilon T})$ is well-defined, that is every subset $S < T$ gives rise to a maximal allocation for S, $[x(t)]_{t\epsilon S}$. Moreover, if $[x(t)]_{t\epsilon S}$ is maximal for S, then we can associate with $[x(t)]_{t\epsilon S}$ a unique "price" vector $p \epsilon R^\ell$, $p >> 0$ such that $\nabla_j U_t(x(t)) = p_j$ if $x_j(t) > 0$ for all $t \epsilon T$ and $\nabla_j U_t(x(t)) \leq p_j$ if $x_j(t) = 0$ for all $t \epsilon T$.

Proof: Max $[\sum_{t\epsilon S} U_t(x(t)) \mid x(t) \epsilon R_+^\ell$ for all $t \epsilon S$ and $\sum_{t\epsilon S} x(t) \leq \sum_{t\epsilon S} I(t)]$ surely has a well-defined solution since the U_t are continuous and we are restricted to a compact set. The second part of the lemma is simply a statement of the necessary first order conditions at a maximum. Since for all j there is some t with $x_j(t) > 0$, $p_j = \nabla_j U_t(s(t)) > 0$ and $p >> 0$. QED.

Lemma 2: If in addition the $[U_t]_{t\epsilon T}$ are concave, then there exists a tuce and a competitive payoff configuration for $(T, V, E, [U_t]_{t\epsilon T})$; the set of tuce is exactly the set of maximal allocations for T. In fact the competitive payoff configuration $[x_t]_{t\epsilon T}$ and the prices p are unique. If the utilities are strictly concave, the tuce is also unique.

Proof: The maximization problem of lemma 1 is now the maximization of a concave function $\sum_{t\epsilon T} U_t(x(t))$ subject to ℓ linear constraints $\sum_{t\epsilon T} x_j(t) \leq \sum_{t\epsilon T} I_j(t)$ for $j = 1 \ldots, \ell$ (which will hold with equality at a maximum due to the monotonicity assumption) and $\ell\,|T|$ nonnegativity constraints. By the Kuhn-Tucker theorem we can find ℓ Lagrange multipliers $p_1 \ldots, p_\ell$ such that at a solution $[x(t)]_{t\epsilon T}$, the Lagrangian

$\sum_{t\in T} U_t(x(t)) - \sum_{t\in T} p_j \sum_{t\in T} (x_j(t)-I_j(t)) = \sum_{t\in T} [U_t(x(t)) - p^t(x(t)-I(t))]$ where $p^t = (p_1,\ldots,p_\ell)$, is also maximized. Of course p is the same as it was in the previous theorem, but now that U_t is concave the first order necessary conditions are also sufficient to assure us that $U_t(x) - p^t(x-I(t))$ is also maximized at $x(t)$ for all $t\in T$. Hence every maximal allocation $[x(t)]_{t\in T}$ is a tuce. In the next lemma we show every tuce is maximal for T. Moreover, since every solution $[x(t)]_{t\in T}$ is clearly regular,

$p_j = \frac{\partial V(T)}{\partial(\sum_{t\in T} I_j(t))}$ for $j=1,\ell$, so p is unique. The competitive payoff configuration is then also unique, since it must be $x_t = \operatorname{Max}_{y\geq 0} U_t(y) - p^t(y-I(t)) = U_t(x(t)) - p^t(x(t)-I(t))$ for all $t\in T$. Finally, if $U_t(x)$ is strictly concave, then so is $[U_t(x) - p^t(x-I(t))]$ as a function of x, hence if $x_t = U_t(x(t) - p^t(x(t)-I(t)) = U_t(\bar{x}(t)) - p^t(\bar{x}(t)-I(t))$, $x(t) = \bar{x}(t)$. QED

<u>Lemma 3</u>: Every competitive payoff configuration for a game $(T,V,E,[U_t]_{t\in T})$ satisfying assumptions 1,2,3a is in the core of $(T,V,E,[U_t]_{t\in T})$. In particular, every competitive payoff configuration is an imputation, hence every tuce $[x(t)]_{t\in T}$ is maximal for T.

<u>Proof</u>: Let $[x(t)]_{t\in T}, p$ be a tuce, so $x(t)$ maximizes $U_t(y) - p^t(y-I(t))$ for all $t\in T$. For any $S\subset T$, $V(S) = \sum_{t\in S} U_t(y(t))$ for some maximal allocation $[y(t)]_{t\in S}$ for S, $\sum_{t\in S} y(t) = \sum_{t\in S} I(t)$. But

$$V(S) = \sum_{t\in S} U_t(y(t)) = \sum_{t\in S} U_t(y(t)) - p^t 0 = \sum_{t\in S} U_t(y(t)) - p^t(\sum_{t\in S} y(t) - \sum_{t\in S} I(t)) =$$

$$\sum_{t\in S} [U_t(y(t) - p^t(y(t) - I(t))] \leq \sum_{t\in S} [U_t(x(t)) - p^t(x(t) - I(t))] = \sum_{t\in S} x_t . \quad \text{QED}$$

Lemma 4: Suppose $E = (t, I(t), >_t)_{t\epsilon T}$ and a representing family $[\bar{U}_t]_{t\epsilon T}$ of concave utility functions satisfy assumptions 1-3a,b,c. Then if $([x(t)]_{t\epsilon T}, p)$ is a competitive equilibrium for the economy E, we can find a family of utilities $[U_t = \lambda_t \bar{U}_t]_{t\epsilon T}$, $\lambda_t > 0$ for all $t\epsilon T$, such that $[x(t)]_{t\epsilon T}$ is a ntuce with the prices p for the game $[T,V,E,[U_t]_{t\epsilon T})$. Furthermore, the unique competitive payoff configuration for $(T,V,E,[U_t]_{t\epsilon T})$ is $([x_t]_{t\epsilon T}, p)$ where $x_t = U_t(x(t)) - p^t(x(t) - I(t)) = U_t(x(t))$ for all $t\epsilon T$.

Proof: By assumption $x(t)$ maximizes $U_t(x)$ on $[x\epsilon R^{\ell}_{+} \mid p^t x \leq p^t I(t)]$. Hence the Kuhn-Tucker theorem assures us that there exists a $\lambda_t > 0$ such that $\frac{\partial \bar{U}_t(x(t))}{\partial x_j} = \lambda_t p_j$ if $x_j(t) > 0$ and $\frac{\partial \bar{U}_t(x(t))}{\partial x_j} \leq \lambda_t p_j$ if $x_j(t) = 0$.

So let $U_t(y) \equiv \frac{1}{\lambda_t} \bar{U}_t(y)$ for all $y \geq 0$. Then look at the function

$\underset{x \geq 0}{\mathrm{Max}}\, U_t(x) - p^t(x - I(t))$. This is a concave function, and since the first order conditions are satisfied at $x = x(t)$, it must be maximized at $x(t)$. But from the monotonicity of $\bar{U}_t$, and hence of $>_t$, we know that $p^t x(t) = p^t I(t)$. QED

Lemma 4b: Conversely, if $[x_t]_{t\epsilon T}$ is a competitive payoff configuration for $(T,V,E,[U_t]_{t\epsilon T})$ and if coincidentally $[x(t)]_{t\epsilon T}$ is an allocation for E such that $U_t(x(t)) = x_t$, then $[x(t)]_{t\epsilon T}$ is a ntuce for $(T,V,E,[U_t])$ and therefore a competitive equilibrium for E.

Proof: By assumption $x_t = \underset{x \geq 0}{\mathrm{Max}}\, U_t(x) - p^t(x - I(t)) \geq U_t(x(t)) - p^t(x(t) - I(t))$. But $\sum_{t\epsilon T} [U_t(x(t) - p^t(x(t) - I(t))] = \sum_{t\epsilon T} U_t(x(t)) = \sum_{t\epsilon T} x_t$, hence $U_t(x(t) - p^t(x(t) - I(t)) = x_t$ for all $t\epsilon T$ and $[x(t)]_{t\epsilon T}$, p is a tuce for $(T,V,E,[U_t]_{t\epsilon T})$. Also it follows that $p^t(x(t) - I(t)) = 0$ for all $t\epsilon T$, hence $[x(t)]_{t\epsilon T}$, p is a ntuce and a competitive equilibria for E. QED

We can summarize the results of lemmas 1-4 in the schematic diagram below:

Unique competitive payoff $[x_t]_{t\in T}$
$\subset$ core $(T,V^1,E,[U_t^1])$

$\boxed{(T,V^1,E,[U_t]_{t\in T})}$

$(t,I(t),>_t)_{t\in T}$

$\boxed{(T,V^2,E,[U_t^2]_{t\in T})}$

$\boxed{E}$

competitive equilibria

$\boxed{(T,V^3,E,[U^3]_{t\in T})}$

$([x^1(t)]_{t\in T},p^1)$

$([x^2(t)]_{t\in T},p^2)$

To each competitive allocation $[x(t)]_{t\in T}$ in E there can be associated a game $(T,V,E,[U_t]_{t\in T})$ such that $[x(t)]_{t\in T}$ is a ntuce and $x_t = U_t(x(t))$, $t\in T$, is the corresponding competitive payoff.

By making yet one more assumption about the representing family of utilities $[U_t]_{t\in T}$ we can complete our characterization of the relationship between the economy E and its derived transferable utility games $(T,V,E,[U_t]_{t\in T})$. If the preferences $>_t$ are strictly convex and if they can be represented by strictly concave utility functions $[U_t]_{t\in T}$ then the maximal allocation $[x(t)]_{t\in T}$ for T with respect to $[U_t]_{t\in T}$ must be unique. For suppose $V(T) = \sum_{t\in T} U_t(x(t)) = \sum_{t\in T} U_t(\bar{x}(t))$ and $\sum_{t\in T} x(t) = \sum_{t\in T} \bar{x}(t) = \sum_{t\in T} I(t)$. Then $\sum_{t\in T} U_t(1/2x(t) + 1/2\bar{x}(t)) > V(T)$ unless $x(t) = \bar{x}(t)$

for all $t \epsilon T$. Hence if the $[U_t]_{t\epsilon T}$ are C^1, monotonic, and strictly concave each game $(T,V,E,[U_t]_{t\epsilon T})$ can be associated with a unique maximal allocation $[x(t)]_{t\epsilon T}$ in E. Thus, $[x_t]_{t\epsilon T}$, $x_t = U_t(x(t))$, is the only imputation in $(T,V,E,[U_t]_{t\epsilon T})$ which arises in the natural way from an allocation in E and the unique competitive payoff configuration for $(T,V,E,[U_t]_{t\epsilon T})$ is given by $[z_t]_{t\epsilon T}$, p where $z_t = U_t(x(t)) - p^t(x(t)-I(t))$ for all $t\epsilon T$ and p is the unique vector of Lagrange multipliers arising from the maximation problem Max $\sum_{t\epsilon T} U_t(y(t))$ such that $\sum_{t\epsilon T} y(t) \leqq \sum_{t\epsilon T} I(t)$, $y(t) \geqq 0$ for all $t\epsilon T$. Moreover $[x(t)]_{t\epsilon T}$ is obviously a pareto optimal allocation for E, for if $[y(t)]_{t\epsilon T}$ were a pareto superior allocation then $\sum_{t\epsilon T} U_t(y(t))$ would be greater than $V(T)$. Finally we demonstrate that not only can we associate with every transferable utility game $(T,V,E,[U_t]_{t\epsilon T})$ a pareto optimal allocation in E, but also to every pareto optimal allocation $[x(t)]_{t\epsilon T}$ in an economy E with convex, monotonic preferences there is a representing family of utilities $[U_t]_{t\epsilon T}$ and a transferable utility game $(T,V,E,[U_t]_{t\epsilon T})$ for which $[x(t)]_{t\epsilon T}$ is maximal for T and a tuce, hence for which $[x_t]_{t\epsilon T} = [U_t(x(t))]_{t\epsilon T}$ is a competitive payoff configuration.

<u>Lemma 5</u>[1]: Suppose $[x(t)]_{t\epsilon T}$ is a pareto optimal allocation in $E = ((t,I(t),>_t)_{t\epsilon T})$ where $>_t$ are representable by utilities satisfying assumptions 3a,b,c,d. Then if the $I(t)$ were set equal to $x(t)$ for all $t\epsilon T$, so we had the game $E' = (t,x(t),>_t)_{t\epsilon T}$, then $[x(t)]_{t\epsilon T}$ would be a competitive equilibrium in E'.

<u>Proof</u>: Let $G_t \equiv [y-x(t) \mid y \gtrsim_t x(t)]$. Let $G = \sum_{t\epsilon T} G_t$. Then since the G_t are convex, so is their sum G. Moreover $0 \epsilon G$ but 0 is not

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*See Debreu <u>Theory of Value</u>.

an interior point, since that would imply the existence of some $z \epsilon G$ with $z << 0$, which is impossible under the monotonicity assumption. Thus we can find a hyperplane separating 0 and G with normal p. If $y >_t x(t)$, then $p^t(y-x(t)) \geq 0$ since $0 \epsilon G_s$ for all $s \neq t$ implies that $y-x(t) \epsilon G$ and so is separated from 0 by the hyperplane normal to p. But we must have $p^t y > p^t x(t)$, for $y - \varepsilon 1 >_t x(t)$ for ε small enough by continuity and the same argument could be used to show that $p^t(y - \varepsilon 1) \geq p^t x(t)$, hence $p^t y > p^t(y - 1) \geq p^t x(t)$. QED

Lemma 6: Let $[x(t)]_{t\epsilon T}$ be a pareto optimal allocation in an economy E with a representing family of utilities $[\bar{U}_t]_{t\epsilon T}$ satisfying assumptions 1,2,3a,b,c,d. Then we can find a game $(T,V,E,[U_t]_{t\epsilon T})$ such that $[x_t]_{t\epsilon T}$, $x_t = U_t(x(t))$ is an imputation. This generalizes lemma 4.

Proof: Consider the economy $E' = (t,x(t),>_t)_{t\epsilon T}$. Since by theorem 5 $[x(t)]_{t\epsilon T}$ is a competitive equilibrium for E', by theorem 4 we can find utilities $[U_t]_{t\epsilon T}$ such that in the game $(T,V',E',[U_t]_{t\epsilon T})$ $[x_t]_{t\epsilon T}$ is a competitive payoff configuration. But since $\sum_{t\epsilon T} x(t) = \sum_{t\epsilon T} I(t)$, it follows that for the game $(T,V,E,[U_t]_{t\epsilon T})$ $[x_t]_{t\epsilon T}$ is an imputation because $V(T) = V'(T)$. This proves the theorem. QED

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